DOCUMENT RESUME

ED 144 805 SE 023 063

AUTHOR Denmark, Tom; And Others
TITLE Final Report: A Teaching Experiment on Equality. PMDC Technical Report No. 6.
INSTITUTION Florida State Univ., Tallahassee. Project for the Mathematical Development of Children.
SPONS AGENCY National Science Foundation, Washington, D.C.
REPORT NO PMDC-TR-6
PUB DATE [76]
GRANT NSF-PES-74-18106-A-03
NOTE 158p.; For related documents, see SE 023 057-058, SE 023 060-066, SE 023 068-072; Contains light and broken type

EDRS PRICE MF-$0.83 HC-$8.69 Plus Postage.
DESCRIPTORS *Cognitive Development; *Educational Research; *Elementary School Mathematics; Grade 1; *Instruction; Learning; *Mathematical Concepts; Primary Education; Teaching Methods
IDENTIFIERS *Equality (Mathematics); *Project for Mathematical Development of Children

ABSTRACT

This is the report of a teaching experiment designed to teach the concept of equality as an equivalence relation to a group of first graders. The rationale, design, selected samples of the instructional materials, summaries of students' performances during the instructional program, analyses of students' performances on a series of evaluations designed to assess their understandings of equality, and recommendations for curriculum planning are included. Students acquired "considerable flexibility" in accepting and interpreting the use of the equals sign in a variety of sentence structures. The equals sign was still viewed primarily as an operator, however, not as a relational symbol. (MS)

**********
* Documents acquired by ERIC include many informal unpublished *
* materials not available from other sources. ERIC makes every effort *
* to obtain the best copy available. Nevertheless, items of marginal *
* reproducibility are often encountered and this affects the quality *
* of the microfiche and hardcopy reproductions ERIC makes available *
* via the ERIC Document Reproduction Service (EDRS). EDRS is not *
* responsible for the quality of the original document. Reproductions *
* supplied by EDRS are the best that can be made from the original. *
**********

ED144805

PMDC Technical Report
No. 6

U S DEPARTMENT OF HEALTH, EDUCATION & WELFARE
NATIONAL INSTITUTE OF EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY

Robert M. Johnson

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC) AND USERS OF THE ERIC SYSTEM."

# Final Report
# A Teaching Experiment on Equality

Tom Denmark, Ella Barco, and Judy Voran

SE 023 063

Portions of this publication may be reproduced without securing permission from the Project for the Mathematical Development of Children (PMDC).

Financial support for the Project for the Mathematical Development of Children has been provided by the National Science Foundation: Grant No. PES 74-18106-A03.

TABLE OF CONTENTS

List of Tables and Figures . . . . . v

Foreword . . . . . vii

Preface . . . . . ix

I. INTRODUCTION TO STUDY . . . . . 1

Background for Study

Purpose of Study

II. THE STUDY . . . . . 5

Description of Population

Design of Study

Log of Activities

III. EVALUATION RESULTS . . . . . 13

Observation Notes

Internal Evaluation

External Evaluations

IV. CONCLUSIONS AND IMPLICATIONS . . . . . 30

Views of Equality

Implications for Curriculum Development

APPENDIX A SUMMARY OF CONCEPTS AND SKILLS, AND SUMMARY OF STUDENT BEHAVIORS . . . . . 39

APPENDIX B SAMPLE OF INSTRUCTIONAL MATERIALS: EXPERIMENTAL TREATMENT . . . . . 57

APPENDIX C INTERNAL EVALUATION POSTTEST . . . . . 141

APPENDIX D EXTERNAL EVALUATION: TEST I . . . . . 149

APPENDIX E EXTERNAL EVALUATION: TEST II . . . . . 153

APPENDIX F EXTERNAL EVALUATION: TEST III . . . . . 157

BIBLIOGRAPHY . . . . . 161

LIST OF TABLES AND FIGURES

TABLES


| | | Page |
|---|---|---|
| 1 | Student Data: Experimental and Control Groups | 8 |
| 2 | Data from Computation Tests: Final Evaluation, Number of Correct Responses | 20 |
| 3 | Scores on Writing/Reading/Total Scales: Evaluation I | 23 |
| 4 | Mann-Whitney U Statistics: Evaluation I | 23 |
| 5 | Total Number of Correct Responses by Sentence Structure: Evaluation I | 24 |
| 6 | Scores on Skill/Understanding/Total Scales: Evaluation II | 25 |
| 7 | Mann-Whitney U Statistics: Evaluation II | 26 |
| 8 | Total Correct Responses by Sentence Categories: Evaluation II | 27 |
| 9 | Scores on Skill/Understanding/Total Scales: Evaluation III | 28 |
| 10 | Mann-Whitney U Statistics: Evaluation III | 29 |
| 11 | Total Correct Responses by Sentence Categories: Evaluation III | 29 |
| 12 | Typical Responses to the Question: Is 2 + 3 = 3 + 1 all right? | 34 |
| 13 | Students' Views of Equality by Categories | 35 |


FIGURES


| | | |
|---|---|---|
| 1 | Modified Ohaus School Balance | 9 |

# FOREWORD

Ed Begle recently remarked that curricular efforts during the 1960's taught us a great deal about how to teach better mathematics, but very little about how to teach mathematics better. The mathematician will, quite likely, agree with both parts of this statement. The layman, the parent, and the elementary school teacher, however, question the thesis that the "new math" was really better than the "old math." At best, the fruits of the mathematics curriculum "revolution" were not sweet. Many judge them to be bitter.

While some viewed the curricular changes of the 1960's to be "revolutionary," others disagreed. Thomas C. O'Brien of Southern Illinois University at Edwardsville recently wrote, "We have not made any fundamental change in school mathematics."[1] He cites Allendoerfer who suggested that a curriculum which heeds the ways in which young children learn mathematics is needed. Such a curriculum would be based on the understanding of children's thinking and learning. It is one thing, however, to recognize that a conceptual model for mathematics curriculum is sound and necessary and to ask that the child's thinking and learning processes be heeded; it is quite another to translate these ideas into a curriculum which can be used effectively by the ordinary elementary school teacher working in the ordinary elementary school classroom.

Moreover, to propose that children's thinking processes should serve as a basis for curriculum development is to presuppose that curriculum makers agree on what these processes are. Such is not the case, but even if it were, curriculum makers do not agree on the implications which the understanding of these thinking processes would have for curriculum development.

In the real world of today's elementary school classroom, where not much hope for drastic changes for the better can be foreseen, it appears that in order to build a realistic, yet sound basis for the mathematics curriculum, children's mathematical thinking must be studied intensively in their usual school habitat. Given an opportunity to think freely, children clearly display certain patterns of thought as they deal with ordinary mathematical situations encountered daily in their classroom. A videotaped record of the outward manifestations of a child's thinking, uninfluenced by any teaching on the part of the interviewer, provides a rich source for conjectures as to what this thinking is, what mental structures the child has developed, and how the child uses these structures when dealing with the ordinary concepts of arithmetic. In addition, an intensive analysis of this videotape generates some conjectures as to the possible sources of what adults view as children's "misconceptions" and about how the school environment (the teacher and the materials) "fights" the child's natural thought processes.

The Project for the Mathematical Development of Children (PMDC)[2] set out

[1]"Why Teach Mathematics?" The Elementary School Journal 73 (Feb. 1973), 258-68.

[2]PMDC is supported by the National Science Foundation, Grant No. PES 74-18106-A03.

to create a more extensive and reliable basis on which to build mathematics curriculum. Accordingly, the emphasis in the first phase is to try to understand the children's intellectual pursuits, specifically their attempts to acquire some basic mathematical skills and concepts.

The PMDC, in its initial phase, works with children in grades 1 and 2. These grades seem to comprise the crucial years for the development of bases for the future learning of mathematics, since key mathematical concepts begin to form at these grade levels. The children's mathematical development is studied by means of:

1. One-to-one videotaped interviews subsequently analyzed by various individuals.

2. Teaching experiments in which specific variables are observed in a group teaching setting with five to fourteen children.

3. Intensive observations of children in their regular classroom setting.

4. Studies designed to investigate intensively the effect of a particular variable or medium on communicating mathematics to young children.

5. Formal testing, both group and one-to-one, designed to provide further insights into young children's mathematical knowledge.

The PMDC staff and the Advisory Board wish to report the Project's activities and findings to all who are interested in mathematical education. One means for accomplishing this is the PMDC publication program.

Many individuals contributed to the activities of PMDC. Its Advisory Board members are: Edward Begle, Edgar Edwards, Walter Dick, Renee Henry, John LeBlanc, Gerald Rising, Charles Smock, Stephen Willoughby and Lauren Woodby. The principal investigators are: Merlyn Behr, Tom Denmark, Stanley Erlwanger, Janice Flake, Larry Hatfield, William McKillip, Eugene D. Nichols, Leonard Pikaart, Leslie Steffe, and the Evaluator, Ray Carry. A special recognition for this publication is given to the PMDC Publications Committee, consisting of Merlyn Behr (Chairman), Thomas Cooney and Tom Denmark.

*Eugene D. Nichols*
Director of PMDC

# PREFACE

This publication is intended to share with the reader information about a teaching experiment designed to teach the concept of equality as an equivalence relation to a selected group of first grade students. The contents of the publication include a rationale for conducting the study, the design of the study, selected samples of the instructional materials, summaries of the students' performances during the instructional program, analyses of the students' performances on a series of evaluations designed to assess their understandings of equality, and suggested recommendations for curriculum planning.

Thanks are due to Max Gerling for the videotaping, to the PMDC administrative assistant, Janelle Hardy, for coordinating the technical aspects of the preparation of this publication, to Maria Pitner for editing the manuscript, and to Joe Schmerler and Julie Rhodes for the typing.

## I. INTRODUCTION TO STUDY

The long range goal for the Project for the Mathematical Development of Children (PMDC) was to build a more thorough and reliable basis for the design and development of a mathematics curriculum. It was the aim of the PMDC staff to formulate a rationale for the construction of a mathematics curriculum based on the ways in which children learn mathematics. The achievement of this goal necessitates a realistic understanding of children's attempts to acquire basic mathematical skills and the understanding of basic mathematical concepts. Thus, during the initial phase of the project (1974-76), the thrust of the various PMDC activities was directed toward obtaining an understanding of the mathematical thinking of young children, ages 5-8. The original long range operational plan for PMDC provided for an expansion of the investigative efforts to other age levels in subsequent years.

The emphasis in PMDC studies was placed on the children's acquisition of knowledge related to topics in the mainstream of the mathematics curriculum--namely--number concepts, numeration (place value), and operations such as addition and subtraction. The scope of some investigative studies, however, was sufficiently broad to cover concepts and skills related to two or more of the above topics. For example, several PMDC investigators (Behr, Denmark, McKillip, and Nichols) investigated children's understanding of mathematical symbolism. The work of these investigators during the first year of the project provided the impetus for this study.

## BACKGROUND FOR STUDY

### Preliminary PMDC Investigations

Two PMDC investigators, Behr and Nichols, conducted open-ended interviews with children in grades 1-6 to determine what meaning, if any, symbolic sentences have to children. In particular, these investigators were interested in how children view the concept of equality. Their findings (Behr, Erlwanger, and Nichols, 1975) indicated that children interpret the equals sign (=) as an operator and not as a relation symbol. That is, children view the equals sign (=) as a symbol which indicates the location of the answer to a problem and not as a symbol which denotes a relation between two numbers.

They found children at each grade level, especially in the lower grades, who could not adequately explain the meaning of sentences such as $5 = 5$ and $1 + 3 = 2 + 2$ in terms of a relation. Regarding the first sentence, a child might explain that $5 = 5$ means (1) that five plus zero equals five, or (2) that five minus zero equals five. In most cases the child would rewrite or change the sentence to show the implied (in their view) operation, for example $5 + 5 = 5$ or $5 - 0 = 5$. Without identifying an operation, implied or written, most children could not ascribe a meaning to a sentence like $5 = 5$. In only a few interviews, and rarely in the lower grades, did they encounter a child who would explain that $5 = 5$ means the numbers are the same.

Children generally tended to view sentences like 1 + 3 = 2 + 2 as the statement of two separate problems. Frequently, children changed the above sentences as follows: 1 + 3 = 4 / 2 + 2 = 4. In some interviews, the child explained that it was wrong to write 2 + 2 after 1 + 3. Their reason was that the answer should be written after the problem. An alternate interpretation of a sentence like 1 + 3 = 2 + 2 was to view it as one problem, changing the symbol = to a +, either mentally or in writing. Under this interpretation, the child would indicate that the answer was 8. Interestingly, some children explained that the sentence could be viewed as both one problem and as two problems. Only a few children indicated by their interview responses that 1 + 3 = 2 + 2 means that 1 + 3 and 2 + 2 are names for the same number.

Denmark found that most first and second graders in his interviews felt that a sentence like 1 + 3 = 4 + 2 was all right. However, they explained that one plus three equals four, ignoring the 2, or they interpreted the sentence as either one problem or two problems as in the discussion above.

The PMDC investigators also observed that most children were not willing to accept written statements of the form a = b + c or a = b - c as being correct. Their reason for rejecting statements in these forms was that the "answer" comes after the problem. Interestingly, some children, who rejected the written statements, accepted oral statements like " six equals two plus four" as being correct, and in some instances used such sentences as they discussed mathematical problems in interview sessions. These observations of children's behaviors are another indication that the typical student apparently learns to interpret the equals sign (=) as a one-directional operator (left-to-right). That is, children do not learn from their experiences in a typical mathematics instructional program to view the equals sign (=) as a symbol which denotes a relation between two numbers, and in particular they do not seemingly detect the symmetrical property of equals through their work in solving equations which are typically presented in any one of the following forms: a + b = c or a - b = c.

The investigative studies by the PMDC investigators were conducted in three schools in two different cities. There were no significant differences in the mathematics curricula for the three schools, and these curricula are representative of the mathematics instruction provided for most children in American schools. There were, however, considerable differences among the student populations as indicated by measures of general ability (IQ), academic achievement, and socioeconomic variables. The factors noted above, together with the fact the observations on children's behavior in interpreting the meaning of the equals sign (=) were the same in all three schools, suggest that the tendency of children to view equality in written statements as an operator is perhaps a universal phenomenon. The PMDC investigators were primarily concerned about young children's concepts of equality; however, studies conducted by other researchers indicate that the conceptualization of the equals sign (=) as an operator is not restricted to young children. Summaries of these research efforts are provided in the following paragraphs.

## Other Research Findings

Considering the importance of written symbolic sentences to the total mathematics curriculum--the written symbolic form is the primary mode for presenting developmental exercises, drill exercises, and evaluation questions--it is somewhat surprising that the effects of symbolic structures on student performance have not been studied more extensively. In one such study Beattie and Deichmann (1972) found that among first and second graders the error rate for horizontal structures is higher than that for vertical structures. These researchers, however, did not report differences in error rates between left and right forms of horizontal sentences, $a \circ b = c$ (left) and $a = b \circ c$ (right). In other studies, for example a study by Grouws (1972), the effects of the location of the place holder (variable) on student achievement have been investigated, but the sentences in these studies were restricted to one form only: $a \circ b = c$. An exception to this general rule was a study conducted by Weaver (1973). In this study Weaver investigated the effects of the location of the place holder on student performance (grades 1-3), but he included in the domain of test questions sentences of the form $a = b \circ c$. By comparing student performances on pairs of symmetrical forms, for example $3 + 2 = \square$ and $\square = 3 + 2$, he found a greater error rate for problems in the form $a = b \circ c$. He also noted that the differences in error rates for the two forms decreased as the grade level increased. The results from his study, however, do not suggest a plausible cause for the differences in the error rates, except that the students participating in the study were generally unfamiliar with sentences of the form $a = b \circ c$.

A study by Renwick, conducted in 1932, provides the most direct support for the validity of observations made by the PMDC investigators. In this study, Renwick working with girls aged 8-14 observed that "most students used the equals sign (=) as a symbol of distinction whose function is to separate a problem from its answer, rather than to bridge two numerically or quantitatively equivalent expressions." Renwick attributed the students' interpretation of the equals sign (=) in this manner to their early training in arithmetic. She also noted that students seemed to prefer the usage of the terms "same" and "alike" to "equals."

The observation that students tend to view the equals sign (=) as a one-directional operator was also noted in a study conducted by Frazer (1976). Frazer provided a group of college freshmen with instruction on the "plus" form of the distributive property: $a \times b + a \times c = a \times (b + c)$; another group with instruction on the "times" form of the property: $ax(b + c) = a \times b + a \times c$; and a third group with instruction on both forms. Her results indicated that students who studied only one form of the property were less likely to transfer the symmetrical property of the generalization to a new application than were students who studied both forms. From these results Frazer concluded that the students who studied only one form of the distributive property generally did not demonstrate an awareness of the symmetrical property of the equals relation.

The results from a 1976 study by Anderson indicate that second graders who receive appropriate instruction on equality as a relation can learn to

interpret the equals sign (=) as a symbol which denotes an equivalence relation between two numbers or expressions. In particular, Anderson noted that students assigned to the experimental treatment were more likely to accept sentences of the forms a = a, a + b = c + d, and a + b = c + a as being correct.

The results obtained from the above studies provide some support for the hypothesis that most students do not develop a concept of equality as an equivalence relation from their experiences in a traditional mathematics curriculum. And further, a probable cause for their failure to do so is the inadequacy of the treatment of equality in most instructional programs.

## Treatment of Equality in Representative Textbook Series

Ten of the most popular elementary mathematics textbook series (student books and teacher's manuals) currently used in elementary schools were examined to obtain one indication of the extent to which equality as a concept is included in elementary mathematics curricula. Based on data obtained from this survey it would appear unlikely that a student would encounter a systematic instructional program which presents equality, in particular an interpretation of the equals sign, as a relation between two numbers. For example, in half of the textbook series examined, including some of the most widely used series, there is no explicit instruction which presents the equals sign as a symbol which denotes a relation between two numbers. In four series there is some attempt to treat equality as a relation, but such instruction is usually restricted to one or two pages at the beginning of the book and in some series there is no instruction on equality at one or more grade levels. In only one textbook series is there evidence that an attempt is made to systematically teach equality as a relation throughout the year at each grade level.

The fact that the vast majority of elementary mathematics textbook series include little, if any, instruction on equality as a relation suggests a very plausible explanation as to why most children do not interpret the equals sign (=) as a symbol which denotes a relation between two numbers. That is, to express the explanation in more vernacular terms, "If the children haven't been taught it, how would we expect them to know it?" Implicit in the acceptance of this explanation is an assumption that if the relational concept of equality is taught to the children, then they are capable of learning it. Since there is no reliable evidence which supports the latter assumption, one must search further for an explanation as to why many children do not learn to view equality, especially when expressed in symbolic notation, as an equivalence relation.

## Summary

Observations of students' behaviors in situations which provided an opportunity for students to express their interpretation of symbolic sentences involving an equals sign suggested to PMDC investigators that the typical student in the primary grades learns to view the equals sign as a

one-directional (left-to-right) operator which connects a problem and its answer. And further, these observations, with support from other studies, indicate that many students in higher grades, even at the college level, do not develop a concept of an equation as an expression of a relation between two numbers. Rather, these older students cling to the notion that an equals sign is a symbol which separates a problem from its answer. This viewpoint may be one possible explanation for the difficulty many students display in a variety of situations such as:

1. Complete 43 = □ + □ ;

2. Complete 8 + 5 = 8 + □ + □ ;

3. Interpret 1 + 2 = 2 + 1.

These observations of students' conceptions of equality raised two significant questions: (1) Is it possible to teach young children to view equality as a relation? and (2) If a student learns to accept equality as a relation, will this enhance the learning of other concepts and skills?

### PURPOSE OF STUDY

The study which is the subject of this report was designed to fulfill the following three purposes:

1. To teach first graders the concept of equality as a relation.

2. To study the effect of understanding and acceptance of the relation concept of equality upon the learning of (a) numeration and expanded notation and (b) the tens in addition.

3. To extend the study to investigate the effort of understanding this concept of equality upon addition and subtraction of two-digit numbers with regrouping.

The decision to conduct the study at the first grade level was based on the fact that it is at this grade level that a student typically encounters symbolic sentences which express the concept of equality.

## II. THE STUDY

### DESCRIPTION OF POPULATION

The study was conducted in an elementary school (K-5) located in Tallahassee, Florida. Approximately 400 students attend this school. The school serves a predominately low socioeconomic community, but about one-third of the students are children of students attending one of two universities located within one mile of the school.

Approximately 100 students were enrolled in the first grade at the

beginning of the school year. These students were assigned to one of three sections, but no attempt was made to group the children homogeneously. The students participating in the study were selected from among those students assigned to one section. This section was selected by the principal of the school.

During the second week of school three tests were administered to each student. The Otis-Lennon Mental Ability Test was a group test. The KeyMath Diagnostic Arithmetic Test and the PMDC Achievement Test: Grade One were administered in individual interview sessions. The data from these tests provided the following information about the students (15 males and 18 females) assigned to this section. The mean IQ was 89. The IQ measured ranged from 58 to 131. The median IQ score was 84, and the standard deviation was 16. The mean total score on the KeyMath test was 34 (G.E. I.O.). The total scores on the KeyMath test ranged from 8 (G.E. less than 0.5) to 73 (G.E. 2.4). The median score was 35, and the standard deviation was 15. The mean total score on the PMDC test was 19.1 (49% correct)*. The total PMDC scores range from 3 (8%) to 37 (95%). The median score was 19.7 (51%); the standard deviation was 9.

## DESIGN OF STUDY

### Overview

The purposes of this teaching experiment were (1) to teach a group of first graders the concept of equality as a relation and (2) to study the effects of such instruction on (a) the learning of expanded notation, (b) bridging tens in addition, and (c) the addition and subtraction algorithms with regrouping. The experimental treatment was provided for a small group, seven students, in order that intensive observations of students' behaviors could be made on a day-to-day basis. A control group was identified to provide a comparison base. Evaluation data were obtained from observation notes, from tests administered by the principal investigator, and from interviews conducted by external evaluators.

### Selection of Students

Seven matched pairs, fourteen students, were selected as subjects for the teaching experiment. The matched pairs were selected to provide a stratified sample of the total class population. The procedures for selecting the stratified sample are as follows:

1. The students were rank ordered according to their total scores on the KeyMath test.

2. The scores were partitioned into three groups. (The regular classroom teachers assigned to students to one of three groups for math

*The mean total score for the PMDC testing population (185 students) was 22.9 (59% correct).

instruction.) The high scores correspond to G.E.'s of 1.5 or above. The middle scores corresponded to G.E.'s between 0.8 and 1.4. The low scores represented G.E.'s of 0.7 or lower.

3. The decision was made to form two matched pairs from both the high and middle groups and three matched pairs from the low group. This selection reflected the distribution of students in the three instructional groups formed by the classroom teacher.

4. Matched pairs were formed by selecting students with approximately the same total KeyMath scores. If three or more students had essentially the same KeyMath scores, the following variables were then considered: sex, IQ, age, and total PMDC scores.

5. After the matched pairs were formed, the regular classroom teacher was asked to evaluate the pairings. With the exception of one pair, middle group, the pairing conformed to the teacher's evaluations of the students' performances in mathematics. Since she had placed one of the students in the pair in the low group, a suitable replacement was found for this student.

6. A student in each matched pair was randomly assigned to the experimental group. The other student was assigned to the control group. The assignment of students to the experimental and control groups was completed by mid-October. From this data until mid-November, when the experimental treatment began, the project staff began collecting observation data on the students and providing special instruction for those students assigned to the experimental group who had not attained a specified set of prerequisite skills. During this interim period, three students assigned to the control group withdrew from school. Replacements for these students were selected from among those students who had not been previously selected to participate in the study.

The final set of matched pairs did not provide the precise matching of students as had been obtained in the original set, but the differences among the variables used in the matching process were small. Data pertinent to the matching process are reported in Table 1 on the following page. The data presented in Table 1 indicate that students in the experimental group generally had slightly higher scores on the two mathematics achievement tests and were older than the students in the control group, but the average IQ for the control students was slightly higher. Both groups included four girls and three boys. The parents of the students constituting Pair 2 were foreign students attending one of the local universities.

## Teaching/Observation Assignments

The principal investigator for this study had the responsibility for teaching the experimental group. Each day during their regular math period these students were taken to a room which was reserved for this study. The students in the control group remained in their regular classroom and

Table 1
Student Data: Experimental and Control Groups

| | Experimental | | | | | Control | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Pair | KeyMath | PMDC | IQ | Sex | Age | KeyMath | PMDC | IQ | Sex | Age |
| 1 | 73 | 35 | 131 | M | 6-8 | 66 | 36 | 124 | F | 6-8 |
| 2 | 55 | 27 | 109 | F | 6-6 | 47 | 25 | 97 | M | 6-6 |
| 3 | 45 | 27 | 109 | F | 6-5 | 45 | 25 | 100 | F | 6-2 |
| 4 | 37 | 20 | 83 | M | 6-8 | 34 | 20 | 97 | M | 5-8 |
| 5 | 26 | 18 | 80 | M | 6-8 | 20 | 15 | 74 | M | 5-11 |
| 6 | 29 | 11 | 72 | F | 5-11 | 27 | 10 | 78 | F | 6-1 |
| 7 | 20 | 12 | 73 | F | 6-3 | 22 | 13 | 84 | F | 5-10 |
| Mean | 40.7 | 21.4 | 93.1 | | 6-5 | 37.3 | 20.6 | 96.3 | | 6-1 |

received their mathematics instruction from the classroom teacher.

Two graduate assistants were assigned to this study for the purpose of obtaining observational data. One graduate assistant observed the experimental group and the other observed the control group. The graduate assistants rotated their observation assignments in two week intervals.

Instructional Materials

Students in both the experimental and the control groups received their mathematics instruction on preliminary concepts and skills in their regular classroom. This instruction was based primarily on the content included in the first forty pages of Holt School Mathematics: Book One. This content included the following concepts and skills.

1. One-to-one correspondence.
2. Counting sets, $N \leq 9$.
3. More, less, one more, one less.
4. Ordering sets.
5. Reading/writing numerals, $N \leq 9$.
6. Empty set.
7. Ordering numbers.
8. Ordinal numbers: first through ninth.

Students in the experimental group did not continue their work in Holt School Mathematics: Book One beyond page 40, since page 41 in this textbook marked the introduction to addition. Students in the control group continued using Holt School Mathematics: Book One throughout the school year.

Students in the experimental group were provided with special instruc-

tional materials. These materials were designed to delay the introduction of most conventional symbolic structures until later in the school year. Specifically, from the time the experimental treatment began, mid-November, until the latter part of April the students worked with symbolic sentences of the form: $a = b \circ c$ or $a \bigcirc b$. During this period the students were introduced to the following topics:

1. Concept of equality.
2. Basic addition facts, sum $\leq 10$.
3. Missing addends.
4. Basic subtraction facts, minuend $\leq 10$.
5. Greater than and less than.
6. Place value, $N \leq 19$.

Detailed lists of specific concepts and skills related to each of the above topics are reported in Appendix A.

The conventional symbolic forms were introduced to the experimental students during the latter part of April. Selected pages from Holt School Mathematics: Book One were used for this instructional unit. Also, sections from Holt School Mathematics: Book One were used during the latter part of the school year to provide the primary instruction on money, time, geometry, fractions, and measurement and to provide supplementary exercises on place value, the hard addition facts, and the addition and subtraction algorithms. Special instructional materials, reflecting the experimental treatment, were also written for later topics. See Appendix A for lists of specific concepts and skills related to these topics.

The development of concepts and skills in the experimental treatment was based on the use of a modified version of the Ohaus School Balance (Figure 1).

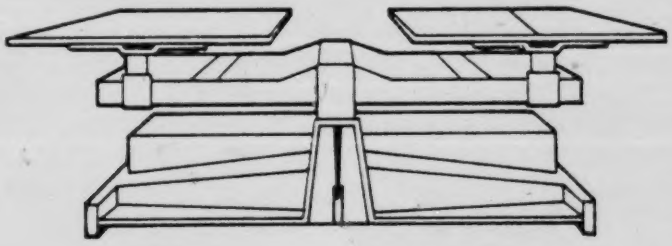


Figure 1
Modified Ohaus School Balance

Modifications of the balances were made by Damon Incorporated. These modifications included:

1. The design of a set of pans, without sides, which were partitioned by color to show either one, two, three, or four regions.

2. The design of a set of metal weights consisting of units (1 x 1 x 1 cm. weighing 3 grams) and tens (1 x 1 x 10 cm. weighing 30 grams).

3. A change in method of attaching the pans to that the scale would be insensitive to the location of the weights on the pans.

4. The addition of a detachable shelf to the back of the scales.

5. An enlargement of the pointer which indicates when the scale is in a balanced position.

6. The attachment of a wooden block above the fulcrum to hold the following signs: $=$, $<$ , $>$ , $\neq$.

The instructional materials written for each lesson in the experimental treatment included the following components:

1. A detailed teacher's guide which contained (a) a statement of the lesson objectives; (b) a list of materials; and (c) a specific sequence of balance demonstrations and related questions.

2. A set of developmental exercises.

3. A set of drill (practice) exercises.

Selected lessons from each of the major units are included in Appendix B.

Evaluation Procedures

Data pertaining to the experimental students' understanding of equality were obtained from three primary sources: observation notes, a test administered by the principal investigator; and test administered by the external evaluators. Assessment data for students in the control group were obtained from two sources: observation notes and tests administered by the external evaluators.

Observation procedures. One graduate assistant observed each experimental lesson. The observer was provided a notebook containing a copy of the instructional materials and blank pages for recording observations. This observer recorded the students' responses and/or reactions to each increment in the developmental component of a lesson. During the practice (drill) component of a lesson, the observer closely observed the work of two or three students, noting questions and comments, errors made, and the strategies used in solving problems. The selection of students for intensive observation was changed each day so that each student was observed at least two times each week. The experimental teacher and the observer met after the lesson to discuss the students' performances during the lesson. Periodically, the experimental class periods were videotaped to obtain additional observational data and to provide an evaluation of the observational notes recorded by the observer. Summaries of the observation notes, by major topics, are contained in Appendix A.

The graduate assistant assigned to the control group observed each student for an extended period at least twice a week. The observer noted the topic being studied, the students' performance on this work, and any instruction provided by the classroom teacher on the meaning of equality. A summary of these notes is provided in Section III.

Internal Evaluation. To supplement the data on the experimental students' interpretations of equality, and in particular their view of this concept in the context of a symbolic sentence, the principal investigator administered a test (Appendix D) on symbolism at the end of the school year. Most of the items on this test required the student to establish a relationship between a symbolic sentence and a balance. Thus, the primary purpose of this posttest was to obtain an indication of the students' conceptualization of equality as related to a balance model. This data therefore provided a base for comparing the students' interpretations of equality in a familiar setting with their view of equality in more neutral situations, that is, the tests administered by the external evaluators.

The posttest was administered in individual interviews by the principal investigator and the graduate assistants. The test was administered in four parts. The times for completing the total test ranged between 45 minutes and one hour. Each interview was videotaped.

External Evaluations. Three assessments of the experimental and control students' interpretations of equality were made by external evaluators, PMDC principal investigators who were not directly involved in the execution of the experimental treatment. These tests (Appendices D, E, and F) were administered in December, March, and June. The external evaluators designed each test, consulting with the principal investigator to determine whether or not specific problems were appropriate in terms of the topics covered in the students' instructional program. The external evaluators administered each test and evaluated the students' performances.

Each test was administered in an individual interview setting. Each interview was videotaped. The average completion times ranged between 15 and 20 minutes.

## LOG OF ACTIVITIES

The preceding sections provided descriptions of the various materials and activities related to the design and conduct of this teaching experiment. This section provides a chronology of important events from the inception of the study in July 1975 to its completion in June 1976. Detailed lists of specific concepts and skills included under each major topic are provided in Appendix A.

July

1. Preparation of a prospectus for a teaching experiment on equality.

2. Approval of the prospectus by the PMDC Advisory Board.

August

1. Selection of project staff.

2. Specifications for modifying Ohaus balance submitted to Damon Incorporated.

3. Arrangements made with a local school for the conduct of the study.

4. Preparation of an outline for the instructional program.

September - Mid-October

1. Students complete PMDC test battery.

2. Students receive instruction on basic number concepts and skills from classroom teacher.

3. Initial selection of experimental and control groups.

4. Instructional materials for the following topics are written: equality, addition, missing addends, and subtraction.

Mid-October - Mid-November

1. Experimental students given prerequisite skills test.

2. Students who have not attained certain prerequisite skills receive appropriate remedial instruction.

3. Students who have attained the prerequisite skills receive instruction which promotes mastery of these skills and/or instruction on topics such as non-metric geometry, time, and graphs. No student instruction directly related to addition, subtraction, or equality.

4. Balances arrive.

Mid-November - Mid-December

1. Experimental treatment begins.

2. Topics covered in this period include: concept of equality and addition, sum $\leq 6$.

3. First assessment by external evaluators.

January

1. Topics covered during this period include: missing addends; subtraction, minuend $\leq 6$.

February

1. Topics covered during this period include: addition, missing addends, and subtraction, sum/minuend $\leq 8$; addition, sum $\leq 10$.

March

1. Topics covered during the first part of this period include: missing addends and subtraction; sum/minuend $\leq 10$.

2. Second assessment by external evaluators.

3. Topics covered during the last part of this period include: time (textbook); greater than and less than.

April

1. Topics covered during this period include: place value, $N \leq 29$; addition, sum $\leq 14$.

2. Students are introduced to conventional symbolic forms (review previous topics by completing exercises in textbook).

3. One student in the experimental group transfers to another school.

May

1. Introduction to sentences of the form $a + b = c + d$.

2. Continued drill on hard addition facts, place value, greater than and less than.

3. Additional topics covered in regular textbook during this period include: time, calendar, money, geometry, and fractions.

4. Students participate in PMDC Spring testing program.

5. Introduction to addition and subtraction with 2-digit addends and minuends.

June

1. Posttest administered by principal investigator.

2. Third assessment by external evaluators.

## III. EVALUATION RESULTS

Information pertaining to the students' concept of equality was obtained from three sources: Observation Notes, a posttest administered by the principal investigator (internal evaluation), and a series of three tests

administered by external evaluators. Data obtained from these assessments are reported below.

## OBSERVATION NOTES

### Experimental Group

Detailed summaries of the students' behaviors are reported by major topics as Appendix A. Changes throughout the school year in the students' perceptions of equality and of symbolic sentences are described below.

1. Prior to the beginning of the experimental treatment, all students had acquired the skills of constructing a set with the same number of members as well as with more members than a given set. All but two students were able to construct sets with less, one more, and one less members than a given set.

2. All students understood the concepts: weighs the same and weighs more (heavier). Also, the students' behaviors indicated that they understood conservation of weight in the following context: If an object is moved from one side of the balance to the other side, the weight does not change.

3. The students readily learned, through their experience in working with a balance, the following generalization: The number of unit weights is the same on each side of the scale if, and only if, the pans balance.

4. At the beginning of the experimental treatment, four students knew the equals sign (=). That is, shown the equals sign they would say, "Equals." None of the students knew the not equals sign ($\neq$).

5. All students learned to read sentences of the forms a = a and a $\neq$ b, and to demonstrate the meaning of such sentences by constructing a correct balance model.

6. All students readily learned to complete sentences of the form a $\bigcirc$ b with either an equals sign (=) or a not equals sign ($\neq$). Three students could complete these sentences without the aid of a balance, after working five or six exercises.

7. All students learned to complete sentences of the forms a = $\square$ + $\square$ and $\square$ = a + b. After some experience in using a balance, the students' behaviors indicated that they verified the correctness of their solutions by comparing the number of units on each side of the balance rather than on the position of the pans--that is, equal if the pans were level, not equal if one pan was higher.

8. In one instance as the students were completing equations of the form: $\square$ = $\square$ + $\square$ two students turned the worksheets around so that the equations were of the form: $\square$ + $\square$ = $\square$. These students

successfully completed this exercise set, but the balance models they constructed were reversed to the form of the equation. For example, they constructed a balance model to show 5 = 3 + 2, but wrote 3 + 2 = 5.

9. With the exception of one student, the students readily learned to solve missing addend problems: a = b + □ and a = □ + b.

10. After some experience in using a balance to solve addition and missing addend equations, the students did not always rely on a strict one-to-one correspondence between the balance model and the equation. For example, to solve 5 = 3 + □ they might set up the balance to show 5 = 2 + 3.

11. The students learned, with some difficulty, to use a balance to solve subtraction sentences: a = □ - □ and □ = a - b. The difficulty did not appear to be with the conceptualization of subtraction (take away). Rather, the difficulty seemed to be with the complexity of the steps involved in solving such problems on a balance. The students, however, quickly learned that the difference set (left side) could be constructed by counting the units remaining on the right side of the balance.

12. As noted above, in the introductory lessons on equality, all students could detect an equals or not equals relation from a balance model. However, only two students were successful in extending these skills to the determination of greater than or less than relations.

13. The students readily accepted the equivalence of 10 units and one ten (long). However, two students needed considerable experience in using tens (longs), before they would consistently use a ten in constructing a set to show a number greater than or equal to ten.

14. The students had no difficulty in learning to complete equations like 14 = □ + □ as 14 = 10 + 4. However, only two students could solve equations like □ = 10 + 4, without the aid of a balance.

15. At the beginning of the experimental treatment, when addition, missing addend, and subtraction equations were first introduced, the students exhibited no difficulty in distinguishing among the three types of sentences. But during the study, as sums and minuends were increased in magnitude in a spiraling curriculum, three students began to make numerous errors which could be attributed to the misreading (non-reading in some cases) of the problem.

16. The students readily accepted the conventional forms of writing addition and subtraction problems: a + b = □, a - b = □, $\underline{a \quad a}$ $\underline{+b}$, $\underline{-b}$. However, in constructing a balance model to solve such problems, they generally constructed the addend sets and the minuend set on the right side of the balance. Three students tended to confuse equations of the form □ + a = b with □ = a + b, and a + □ = b

with a = □ + b; but when asked to read aloud the sentence they interpreted such sentences correctly.

17. Except for one student, the students experienced considerable frustration in learning to interpret and to model sentences of the form a + b = c + d. Their frustration seemed to be due to their tendency to interpret such sentences as a single problem to be solve, rather than as a relationship between two problems. By using 2-region pans on each side of a balance the students eventually learned to complete equations like a + b = c + □ and to determine whether equations of the form a + b = c + d were correct or not correct. Based on observations of their work, it was evident that the students were aware of the fact that the balance showed an equals relation if and only if the number of units on both pans were the same. However, only one student exhibited evidence of an awareness of a relation between two problems.

18. The prospectus for this teaching experiment on equality included a study of the students' concepts of equality in the context of learning the addition and subtraction algorithms, with regrouping. This aspect of the study was not undertaken because the majority of the students did not progress to the point that they were ready to learn these skills in the time allocated to the experimental treatment. However, the students, except two, learned to add and subtract 2-digit numbers without regrouping.

The above remarks on the observations of the students' behaviors suggest that the students learned to view the equals sign (=) as a symbol which indicates a sameness of number relation. In particular, a balance shows an equals relation if and only if each side of the balance has the same number of units (a ten was accepted as being equivalent to 10 units). However, with the exception of one student, their view of equality was in a sense one-directional, regardless of the written form of the equation. That is, in addition to denoting a sameness of number relation between a problem and its answer, the equals sign was also a symbol which identified (pointed out in a sense) the location of the answer, either on the left or the right side of the equation. Only one student exhibited evidence of viewing the equals sign as a symbol which denotes an equivalence between two expressions, thus indicating that the two expressions are names for the same number.

Control Group

Students in the control group were assigned to one of three groups for mathematics instruction. The students were grouped on the basis of achievement in mathematics. At the beginning of the experimental treatment, mid-November, two of the control students were in the high group; two were in the middle group; and three were in the low group. After the Christmas break, one student was moved from the middle to the high group, and one student in the low group was moved to the middle group. Thus, for most of the school year the distribution of control students among the three groups was as follows: 3-high, 2-middle, and 2-low.

The primary source of instruction for all three groups was Holt School Mathematics: Book One. The classroom teacher provided supplementary exercises through boardwork, worksheets, games, and some work with manipulative aids. By the end of the year, the high and middle groups had essentially covered all of the textbook. This instruction included work through addition and subtraction with 2-digit numbers, without regrouping. The high group received some supplementary exercises, boardwork in particular, which was directed toward the development of a mastery of the concepts and skills taught. The work for the students in the low group was generally restricted to easy addition and subtraction problems, sums/minuends less than or equal to 10.

None of the students in the three groups received explicit instruction related to the development of equality as an equivalence relation. The students were told that the symbol = was an equals sign, but they were encouraged to read equations like 3 + 2 = 5 as "Three plus two is the same as five." In the process of being introduced to the vertical forms, $\begin{array}{r} a \\ +b \end{array}$ and $\begin{array}{r} a \\ -b \end{array}$, the students were told that the line under the second number stood for the equals sign. Otherwise there was no specific instruction on the equals sign.

The students in the control group worked almost exclusively with equations of the form a ∘ b = c. An exception to this practice occurred when the students were studying different names for a number. In the introduction to this lesson to the high and middle groups, the teacher wrote, for example, 6 = on the board and asked the students to complete the sentence in as many ways as they could. The students had no problems in interpreting the meaning of these questions. One student wrote 6 = 6, but when the teacher asked the student to write something else, the student wrote 6 = 6 + 0.

## INTERNAL EVALUATION

The students in the experimental group* were administered a test, developed by the principal investigator, during the last week of school. This test was designed to assess the following: (1) the students' proficiency in computing addition and subtraction facts, (2) the students ability in using a balance to solve problems, (3) the students' ability in interpreting a balance model, and (5) the students' knowledge of 2-digit numbers. This test was administered in four parts, each in an interview setting which was videotaped. A copy of the test is presented in Appendix C.

The results from the posttest administered by the principal investigator provided the following information.

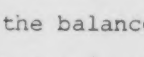
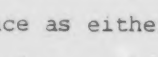
1. All students demonstrated that they could read 2-digit numerals, $n < 30$ and could use the tens rods and ones cubes in constructing sets on the balances. All but two students demonstrated that they

*One student in the original experimental group transferred to another school after the spring break, mid-April.

could count dimes and pennies up to 30 cents.

2. Four of the six students demonstrated that they knew the meaning of both more and less.

3. Four of the six students demonstrated that they could complete sentences of the form a ○ b + c and a + b ○ c by using = or ≠ in the circle. However, only one of these students exhibited evidence that the position of the pans (level or not level) was related to whether equals or not equals was the appropriate relation. The other students based their answers on a relationship between numbers after they computed the sum of two numbers.

4. All students demonstrated that they could use a balance to solve equations of the following forms: a + b = □; □ = a + b; a - b = □; □ = a - b; a = □ + b; a = □ + □; a + b = c + □. Some students, however, were not successful in showing the solutions to some problems. It was observed that in each case where an error occurred, the student had not placed the pans on the balance to establish an appropriate one-to-one correspondence between the regions on the pans and the equation. For example, to solve 5 + 4 = □, four students set up the balance as either  or  . These students put 5 units on the left side of the balance and 4 units on the right side. Three of these four students added a fifth unit to the right side and wrote 5 + 4 = 1. The other added 5 and 4 on her fingers and wrote 5 + 4 = 9.

Altogether there were 10 problems for which the student had to place pans on the balance before solving the equation. Thus there was a total of 60 responses (six students/10 problems). The following matrix shows the relationship between the way the students placed the pans on the balance and their subsequent demonstrated proficiency in using the balance to solve an equation.

| | | Appropriate placement of pans: Yes | Appropriate placement of pans: No |
|---|---|---|---|
| Successfully showed solution to equation on balance. | Yes | 38 | 11 |
| | No | 0 | 11 |

The data in the above matrix suggest that an appropriate placement of pans on the balance enabled the student to demonstrate successfully the solution of the equation on the balance. However, with an inappropriate placement of pans, the students demonstrated acceptable solutions in only 50% of the cases.

It should be noted that during the instructional phase of the study the students did not experience situations in which they had to place pans on the balance before solving an equation.

5. There were four items on the final test in which the pans were appropriately placed on the balance by the interviewer before the student was asked to solve an equation. In two cases the equations were of the form a = □ + □, and in two cases the equations were of the form □ = a + b.

   In all four cases each student was successful in using the balance to solve the equation.

6. There were two opportunities for the students to construct a balance model of an equation of the form a = b + c. In each case the interviewer had appropriately placed the pans on the balance.

   All six students constructed successfully constructed models for both equations.

7. There were five items on the test for which the student was asked to describe the model which had been set up on the balance. Only an oral response was required in two cases; in the other three cases the students were asked to write a sentence to describe the model.

   Only two students successfully completed all five items. Two other students demonstrated success on two of the five items, and the other two students were unsuccessful on all five items.

   The most frequent error (oral description) was the mere recitation of numbers shown without any reference to an operation or to the equals relation. For example, the student said, "twelve, seven, five" when the balance was set up to show 12 = 7 + 5. In two instances, the student reversed the order of the operation and the equals relation in the oral description. For example, "Twelve plus seven equals five."

   In one item the balance was set up to show 5 = 3 + 2. Two students, who were not successful on all five description items, wrote the correct equation. The other two students wrote 5 + 3 = 2.

   When the balance was set up to show 4 = 4 or 3 ≠ 5, the four students who were not successful on all five description items wrote, in each case, an equation or an expression which involved an operation. Typical responses were of the form 4 + 4, or 4 + 4 = 8, or 4 = 4 + □. Interestingly, the two students, who wrote acceptable equations, wrote an equation which involved an operation. For example, instead of writing 4 = 4, one student wrote 4 - 0 = 4, and rather than writing 3 ≠ 5 or 3 < 5, one student wrote 3 ≠ 3 + 2.

8. Five of the six students exhibited evidence that they could relate a balance model which had been set up to show a given equation to a

second equation. For example, given the situation where the balance had been set up to show 12 = 7 + 5, the student could modify the given model to show 11 = 6 + 5 by removing one unit from the set of 12 and one unit from the set of 7. Or in another case given the model to show 11 = 6 + 5, to show 11 = 5 + 6 the student interchanged the positions of the sets of 6 and 5 or simply observed that the given model also represented the second equation.

9. All students demonstrated an ability to read equations of all types: addition, subtraction, missing addend.

10. All students demonstrated a high degree of proficiency in computing basic addition and subtraction facts, presented in either horizontal or vertical form. As part of the final evaluation the students were given three computation tests: easy addition facts, easy subtraction facts, hard addition facts. The students had the option of using their balance to answer problems on each test, but none of the students elected to use a balance to compute easy facts. The results on the tests of basic facts are reported in Table 2.

Table 2
Data From Computation Tests: Final Evaluation
Number of Correct Responses

| Student | Easy Addition (15) | Easy Subtraction (15) | Hard Addition (6) | Total (36) |
|---|---|---|---|---|
| 1 | 15 | 15 | 6 | 36 |
| *3 | 15 | 15 | 6 | 36 |
| 4 | 15 | 15 | 6 | 36 |
| 5 | 15 | 13 | 4 | 32 |
| 6 | 15 | 15 | 5 | 35 |
| 7 | 15 | 13 | 6 | 34 |
| mean | 15 | 14.3 | 5.5 | 34.8 |

*Student 2 did not take this test.

Errors on the easy subtraction test were all related to problems involving zero as the subtrahend or difference. One student made two errors of the following type: n - 0 = 0. This student was also making the same error when demonstrating how to use the balance (comment 4 above) to solve □ = 6 - 0. Another student made two errors by applying the following generalization: n - n = n. However, this student was successful in using the balance (comment 4 above) to solve 6 - 6 = □.

The following conclusions, pertaining to the experimental students' understanding of equality, their understanding of symbolic sentences, their understanding of the relationship between sentences and balance models, and

their computational skills were derived from the above data.

1. All students in the experimental group demonstrated by their performance on the posttest (internal evaluation) that they had learned: (a) to compute sums and differences with or without the aid of a balance, (b) to read equations, (c) to count and construct sets with more than 10 members, and (d) to read 2-digit numerals.

2. Most, but not all, students had learned: (a) the meaning of more and less; (b) efficient techniques of constructing a new balance model by modifying an existing balance model, thus exhibiting some evidence of being aware of relationships between two equations; and (c) to complete equations of the form a ○ b + c by writing = or ≠ in the ○ .

3. Only two students exhibited evidence that they understood the relationship between a balance model and the corresponding equation. That is, these students seemed to comprehend that the use of the term equals, whether in the context of describing a balance model or in the context of an equation, meant that the numbers represented on both sides of the balance or equation were the same. However, these students were unable to use equals in a passive sense, for example, to state that 4 equals 4. Rather, they interpreted equals only in the context of an action situation; for example, 4 take away 0 equals 4. The other four students exhibited no evidence of ascribing a meaning to the term equals. For these students the equals sign was merely a symbol which separated a problem and the answer to the problem and did not signify any other meaningful relationship between these two entities. It was noted, however, that these four students were aware of the fact that when the balance was level or the answer was correct, there would be the same number of units on both sides of the balance.

## EXTERNAL EVALUATIONS

An assessment of the effects which the experimental treatment had on the students' conceptualization of equality was made by a team of PMDC evaluators who were not directly involved in the conduct of this teaching experiment. The external evaluation consisted of a series of three tests. The first test was administered toward the beginning of the study (December), the second test was given approximately at the mid-point of the study (March), and the third assessment was made after the experimental treatment was terminated (June). Each test in the series was an individual interview, lasting on an average between 15 and 20 minutes.

A script was prepared to guide the evaluators in the conduct of each interview, providing a degree of standardization in the administration of each test. The evaluators, however, exercised some freedom in asking probing questions in those cases where, in the evaluator's judgment, the prescribed stimulus did not elicit a response which provided a valid indication of a student's understanding of the concept or skill being assessed. Each interview was videotaped; thus, variations from the standard script could be taken

into consideration in the final analysis of the student's performance.

The evaluators viewed each videotape in the process of scoring the students' performances. A scoring guide was prepared for each test to assist the evaluators in making their assessments. Prior to the scoring of each test the evaluators jointly scored the performances of two or three students in order to reach an agreement on the assignment of points to particular behaviors. Also both evaluators independently assessed the performances of several students. These analyses were compared to assess the reliability of the scoring procedures.

Copies of the three tests are provided in Appendices D, E, and F. The results obtained from these evaluations are reported in the paragraphs which follow.

First Evaluation

The first test consisted of two parts. Items in the first part were administered as follows:

1. The student was shown an equation of the form a = □ + □, □ = a + b, a ○ b + c, □ + □ = a, a + b = □, and a + b ○ c.
2. The student was told to write something in the □'s or ○ to make it right.
3. The student was then asked to read the sentence.
4. The student was asked if the sentence was okay.

There were ten items in the first part of the test.

The second part of the test was composed of four items. In each item the student was shown an equation of the form a = a, a = b + c, or a + b = c + d. The student was asked to read the equation and then was asked to tell whether or not the equation was okay.

The primary purposes of this test were (1) to obtain an indication of the students' skills in completing open sentences (writing) and (2) to obtain an indication of the students' skills in reading mathematical sentences. One point was awarded for each open sentence correctly completed, and one point was awarded for each sentence correctly read. No measure was assigned to the question, "Is it okay?" The maximum writing score was 10 and the maximum reading score was 14. The maximum total score was 24. Data pertaining to the students' writing and reading skills from the first test are reported in Table 3 on the following page.

As previously noted, the primary purposes of the first external evaluation were to determine the effects of the experimental treatment on the acquisition of skills pertaining to the completion (writing) and reading

Table 3
Scores on Writing/Reading/Total Scales: Evaluation I

| Pair | Writing | | Reading | | Total | |
|---|---|---|---|---|---|---|
| | Experimental | Control | Experimental | Control | Experimental | Control |
| 1 | 10 | 9 | 14 | 8 | 24 | 17 |
| 2 | 9 | 9 | 14 | 7 | 23 | 16 |
| 3 | 9 | 6 | 14 | 11 | 23 | 17 |
| 4 | 4 | 4 | 9 | 8 | 13 | 12 |
| 5 | 4 | 7 | 6 | 0 | 10 | 7 |
| 6 | 4 | 2 | 14 | 0 | 18 | 2 |
| 7 | 1 | 5 | 7 | 6 | 8 | 11 |
| mean | 5.4 | 5.7 | 10.9 | 5.7 | 16.3 | 11.6 |

of mathematical sentences. The following three null hypotheses were tested to determine such effects.

$H_1$: There is no significant difference between the mean writing scores of the experimental and control groups.

$H_2$: There is no significant difference between the mean reading scores of the experimental and control groups.

$H_3$: There is no significant difference between the mean total scores of the experimental and control groups.

Each null hypothesis was tested against an alternate hypothesis which stated that the experimental group would have a higher mean score.

The Mann-Whitney U Test, a nonparametric distribution-free rank test, was used to make the analyses. The Mann-Whitney U statistics are reported in Table 4.

Table 4
Mann-Whitney U Statistics: Evaluation I

| Scale | U | p |
|---|---|---|
| Writing | 27 | .402 |
| Reading | 8 | .019 |
| Total | 13 | .082 |

$n_1=7$, $n_2=7$, $\alpha = .05$

The data in Table 4 indicate that the effects of the experimental treatment were significant, $P \leq .05$, in the area of reading skills only. An analysis of the students' responses, by categories of sentence structures, indicated that whereas the students in both groups had acquired the same level of proficiency in completing open sentences, the students in the experimental group were more skillful in reading sentences which were not in the form a + b = c. Data from the item analysis are reported in Table 5.

Table 5

Total Number of Correct Responses by Sentence Structure: Evaluation I

| Sentence Structure | Writing | | Reading | |
|---|---|---|---|---|
| | Experimental | Control | Experimental | Control |
| a = b + c | 21 | 21 | 37 | 7 |
| a + b = c | 20 | 21 | 24 | 22 |
| a = a | — | — | 6 | 3 |
| a + b = c + d | — | — | 10 | 8 |

The data reported in Table 5 reveal several interesting patterns in the students' response. First, it should be noted that students in the control group were considerably more successful in completing sentences of the form a = b + c, than they were in reading sentences of this type. Observations of the control students' behaviors on reading tasks indicated that some of these students either refused to read the sentence or they reversed the order of the numbers. Thus, for example, reading 3 = 2 + 1 as "one plus two is the same as three." These behaviors suggest that whereas the students were capable of interpreting an unfamiliar symbolic sentence as a problem to be solved, these same students had also acquired, early in their school experience, the notion that a particular order must be preserved in reading (orally) a sentence; namely, the problem precedes the answer.

The fact that the experimental students were considerably more successful in reading sentences of the forms a = b + c and a = a is not surprising. These sentences were familiar to the experimental students. It is, however, interesting that the experimental students were slightly more successful in reading sentence structures of the form a + b = c and a + b = c + d, forms which they had not encountered in their instructional program. This suggests that initially the experimental treatment enabled the students to develop greater flexibility in reading skills. That is, the experimental treatment did not promote the acquisition of a mind set that there is a prescribed order in the expression of a relation in symbolic forms. It should be noted, however, that one student in the experimental group, the weakest student in terms of ability and achievement, consistently changed each sentence of the form a + b = c to the form a = b + c.

Second Evaluation

The second test administered by the external evaluators consisted of fourteen items. The following procedures were used with each item:

1. The student was shown open sentences of the forms $a + b = \square$, $\square = a + b$, $a - b = \square$, $\square = a - b$, $a + \square = b$, $a = b + \square$, $\square + \square = a$, $a = \square + \square$, and $a + b = \square + \square$.

2. The student was told to work the problem--the student wrote the response.

3. The student was shown four manipulative aids: sticks, unifix cubes, beans, blocks.

4. The evaluator pointing to the manipulatives said, "Use one of these to show me what the problem says."

The items on this test were scored awarding one point for a correct completion of the open sentence. Two points were given for an acceptable demonstration of the problem with a manipulative; one point was awarded for a partial demonstration. For example, to show 4 = 2 + 2 if the student made only two sets, each with 2 beans, and provided no other evidence of relating these sets to the number 4, the response was scored by awarding only one point.

The scoring procedures used with the second test yielded three scores: skill (computation), understanding, and total. Data on the students' performances by scales are reported in Table 6.

Table 6

Scores on Skill/Understanding/Total Scales: Evaluation II

| Pair | Skill | | Understanding | | Total | |
|---|---|---|---|---|---|---|
| | Experimental | Control | Experimental | Control | Experimental | Control |
| 1 | 14 | 12 | 28 | 24 | 42 | 36 |
| 2 | 10 | 11 | 20 | 9 | 30 | 20 |
| 3 | 11 | 12 | 16 | 15 | 27 | 27 |
| 4 | 8 | 13 | 10 | 8 | 18 | 21 |
| 5 | 5 | 2 | 9 | 0 | 14 | 2 |
| 6 | 2 | 4 | 2 | 2 | 4 | 6 |
| 7 | 2 | 2 | 4 | 4 | 6 | 6 |
| mean | 7.4 | 8.0 | 12.7 | 8.4 | 20.1 | 16.9 |

The data reported in Table 6 were used as a basis for testing the following null hypotheses:

$H_4$: There is no significant difference between the mean skill scores of the experimental and control groups.

$H_5$: There is no significant difference between the mean understanding scores of the experimental and control groups.

$H_6$: There is no significant difference between the mean total scores of the experimental and control groups.

Each null hypothesis was tested against an alternate hypothesis which stated that the experimental group would have a higher mean score.

The Mann-Whitney U Test was used to make the analyses. The U Statistics are reported in Table 7.

Table 7
Mann-Whitney U Statistics: Evaluation II

| Scale | U | P |
|---|---|---|
| Skill | 21 | .355 |
| Understanding | 15 | .130 |
| Total | 21 | .355 |

$n_1=7$, $n_2=7$, $\alpha=.05$

Since the p-value for each scale was greater than .05, none of the null hypotheses were rejected.

Although there were no statistically significant differences between the mean scores on the various scales, an analysis of the responses by categories of sentences revealed several marked differences in the students' interpretation of certain classes of sentences. Data on the total number of correct responses for each sentence category are reported in Table 8 on the following page. The data reported in Table 8 show that students in the control group were more successful in solving missing addend problems, forms a = b = □ and a + □ = b. It was previously noted in the discussion of student behaviors under Observation Notes, that students in the experimental group tended to confuse the structure of missing addend sentences with that of an addition sentence. The data obtained from the second external evaluation support this observation. Students in the experimental group were somewhat more successful in completing sentences of the form a + b = □ + □. The students' abilities

Table 8

Total Correct Responses by Sentence Categories: Evaluation II

| Sentence Category | Skill | | Understanding | | | |
|---|---|---|---|---|---|---|
| | Experimental | Control | Experimental | | Control | |
| | | | Partial | Complete | Partial | Complete |
| a = b + c | 7 | 5 | 1 | 5 | 1 | 1 |
| a + b = c | 6 | 6 | 0 | 5 | 4 | 2 |
| a = b - c | 5 | 5 | 0 | 5 | 0 | 1 |
| a - b = c | 5 | 6 | 0 | 4 | 1 | 3 |
| a = b + □ | 5 | 9 | 0 | 5 | 2 | 4 |
| a + □ = b | 3 | 8 | 0 | 3 | 4 | 4 |
| a = □ + □ | 8 | 9 | 2 | 5 | 5 | 2 |
| + = a | 7 | 7 | 1 | 6 | 3 | 4 |
| a + b = □ + □ | 6 | 3 | 2 | 4 | 0 | 2 |

in completing sentences of the other forms were approximately the same for both groups. An interesting observation from these data is that the performances of the control students were seemingly unaffected by the structure of the equation.

Students in the experimental group were generally more successful in using a manipulative aid to show the meaning of an equation. This observation is probably attributable to the fact the students in the experimental group had considerably more experience in using a manipulative aid to model an equation. In particular, they were taught to construct a set for each number in an equation. Thus, for example, to show the equation 6 = 4 + 2 they would most likely construct both addend sets and the sum set, thereby receiving full credit for the demonstration, rather than construct only the addend sets and receive partial credit.

## Third Evaluation

There were fourteen items on the third test administered by the external evaluators. These items were of three basic types. The first eight items were administered according to the following procedures:

1. The evaluator wrote on a piece of paper an equation of the form a = b, a = b + c, a + b = c, a + b = c + d.

2. The student was asked to read the equation.

3. The student was asked if the equation was okay.

4. The student was asked to tell why the equation was okay or why it was not okay.

5. The student was given a pile of beans and was asked to show with the beans what the equation says. This step was omitted in items 7 and 8.

In the next three items an equation of the form a = b + c or a + b = c + d read to the student and then the student was asked if it was okay. The last three items were missing addend problems. Two were written, a = b + □ and a + □ = b, and one was a story problem, presented orally.

In scoring this test the evaluators awarded one point for each equation read correctly; one point for correct response to the question, "Is this okay?"; one point for each correct explanation of why the equation was or was not okay; two points for an acceptable model for an equation; one point for a partial model; and one point for the correct answer to a problem. This scoring procedure yielded data for three scales: skill (reading), understanding, and total. Data for these scales are reported in Table 9.

Table 9
Scores on Skill/Understanding/Total Scales: Evaluation III

| Pair | Skill | | Understanding | | Total | |
|---|---|---|---|---|---|---|
| | Experimental | Control | Experimental | Control | Experimental | Control |
| 1 | 11 | 8 | 25 | 10 | 36 | 18 |
| 2* | - | - | - | - | - | - |
| 3 | 10 | 11 | 16 | 31 | 26 | 42 |
| 4 | 7 | 10 | 22 | 17 | 29 | 27 |
| 5 | 9 | 0 | 20 | 4 | 29 | 4 |
| 6 | 9 | 4 | 15 | 7 | 24 | 11 |
| 7 | 7 | 4 | 6 | 5 | 13 | 9 |
| mean | 8.8 | 6.2 | 17.3 | 12.3 | 26.2 | 18.5 |

*One student in Pair 2 transferred to another school.

The data reported in Table 9 were used in testing the following hypotheses:

$H_7$: There is no significant difference between the mean skill scores of the experimental and control groups.

$H_8$: There is no significant difference between the mean understanding scores of the experimental and control groups.

$H_9$: There is no significant difference between the mean total scores of the experimental and control groups.

Each null hypothesis was tested against an alternate hypothesis which stated that the experimental group would have a higher mean score.

The Mann-Whitney U Test was used in testing each hypothesis. The pertinent U-Statistics are reported in Table 10.

Table 10
Mann-Whitney U Statistics: Evaluation III

| Scale | U | P |
|---|---|---|
| Skill | 11 | .155 |
| Understanding | 11 | .155 |
| Total | 10 | .120 |

$n_1 = 6$, $n_2 = 6$, $\alpha = .05$

The p-value for each hypothesis was greater than .05, therefore none of the null hypotheses was rejected in favor of an alternate hypothesis.

The data reported in Table 9 show that the mean scores for the skill and understanding scales were higher for the experimental group, but not significantly higher, than the means for the control group. An analysis of the students' responses by categories of equations reveals several response patterns which account for most of the differences in mean scores. The data on student responses by sentence categories are reported in Table 11.

Table 11
Total Correct Responses By Sentence Categories: Evaluation III

| Sentence Category | Skill | | Understanding | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | Experimental | | | Control | | |
| | Experimental | Control | Okay | Meaning | Model | Okay | Meaning | Model |
| a = a | 6 | 4 | 1 | 1 | 4 | 4 | 1 | 3 |
| a = b + c | 12 | 7 | 13 | 3 | 12 | 12 | 2 | 2 |
| a + b = c | 11 | 9 | 12 | 2 | 13 | 13 | 4 | 2 |
| a + b = c + d | 17 | 10 | 16 | 3 | 9 | 9 | 3 | 6 |
| a = b + □ | 1 | 3 | - | - | - | - | - | - |
| a + □ = b | 6 | 7 | - | - | - | - | - | - |

The data reported in Table 11 indicate that students in the experimental group exhibited more skill in reading sentences of the forms a = b + c and

$a + b = c + d$. These results are not surprising, since students in the control group had not encountered such sentences in their instructional program. Otherwise there were only small differences in the reading skills for the two groups.

It is of especial interest to note that only one student in the experimental group accepted $5 = 5$ as a correct sentence. All of the other students insisted that the sentence should involve an operation. Thus, they explained that $5 + 0 = 5$ or $5 = 5 + 0$ would be all right. Students in the experimental group were more willing to accept sentences of the form $a + b = c + d$, and they were more successful in constructing a model for an equation. Both of these behaviors are a reflection of the content of the experimental treatment.

## IV. CONCLUSIONS AND IMPLICATIONS

One of the startling findings of the PMDC investigation into the mathematical thinking of young children was the verification of the fact that students, especially those in the 5-8 year old age range, tend to interpret the equals symbol (=) not as a symbol which denotes a relationship between two numbers, but as an operator symbol which connects a problem and its answer. Furthermore, most students acquire the notion that the operator is one-directional, left-to-right. That is, the problem must precede the answer. These conclusions were derived from observations of students' behaviors in situations where, for example, they rejected $5 = 5$ as being okay and then rephrased the sentence to include an operation ($5 + 0 = 5$), and rejected $4 = 3 + 1$ in favor of the conventional form $3 + 1 = 4$.

One explanation for the students' rejection of sentences of the forms $a = a$, $a = b \circ c$ and $a \circ b = c \circ d$ is that the students' mathematical training had provided, at most, very limited experiences with sentences of these forms. Therefore the students' reluctance to accept the unfamiliar sentence forms could be merely a reflection of the limited scope of their mathematical instruction. This line of reasoning provides a reasonable explanation for the students' behaviors, but it does not take into consideration one significant fact; namely, that the students had not learned from their mathematical instruction to view the equals sign as a symbol which denotes a relation between two numbers. Implicit in this explanation is an assumption that if students are provided suitable experiences with sentences of the forms $a = b$, $a = b \circ c$, and $a \circ b = c \circ d$, as well as experiences with the form $a \circ b = c$, then students would view the equals sign as a relation and not as an operator. PMDC investigators found no evidence in the existing literature to support this assumption. In fact, the available evidence suggests that many students, after several years of mathematical training, which includes experiences with various forms of equality statements, still cling to the notion of the equals sign as an operator, and not as a symbol which relates two names for the same number. Since the concept of equality as a relation is essential for students' understanding of other mathematical concepts, as presented in current textbooks, and since students begin to develop their concept of equality in the first grade, the teaching experiment discussed in this publication was designed to investigate the following research question:

Will students who are provided mathematical instruction encompassing sentences of the forms $a = a$, $a \circ b = c$, $a = b \circ c$, and $a \circ b = c \circ d$ in their first year of school learn to view the equals sign as a symbol which denotes a relation between two names for the same number?

Data obtained during the study from observation notes and formal assessments, internal and external, clearly indicate that all students in the experimental treatment acquired considerable flexibility in accepting and interpreting the use of the equals sign in a variety of sentence structures. It was also observed that some students in the control group who were taught to read the equals sign as "the same as" acquired the same flexibility of thought. But the data indicate that only two students (one experimental and one control) viewed the equals sign as a relational symbol at the end of the year. The behaviors of most of the other students suggest that they viewed the equals sign as an operator connecting a problem with its answer or that they attached little, if any, significance to the use of an equals sign in the context of a symbolic sentence. Precise descriptions of the students' views of equality are provided below.

## VIEWS OF EQUALITY

Analyses of the data obtained from this study of children's conception of equality suggest the existence of five distinct interpretations of equality among the first grade students, experimental and control group. That is, the students' behaviors (responses to questions, oral explanations, and written responses) can be clustered into five categories (0-4), each representing a different view of equality. These views of equality range from those cases in which a child's behaviors indicate no discernible interpretation of equals, to those cases in which a child's behaviors indicate that the equals sign is a symbol which denotes that two expressions are names for the same number. The five clusters of behaviors are described below.

Category 0. The child's behaviors suggest that the child does not have any comprehension of what the equals sign means. The following examples are typical of the behaviors classified in this category:

a. The child reads $\square = 4 + 2$ correctly, writes $1 = 4 + 2$, and says that it is okay because "There is a 4 here and a 2 here."

b. The child completes $5 \bigcirc 3 + 2$ as follows: $5 \textcircled{4} 3 + 2$.

c. The child completes $3 + \square = 3$ as $3 + \boxed{3} = 3$, and says "3 plus 3 plus 3 is 9."

Category 1. The child's behaviors suggest that the child recognizes that an equals sign is a symbol which should be included (usually in a specific location) in a written sentence but ascribes no specific meaning to the symbol. Behaviors typical of

those assigned to this category are as follows:

a. The child states that 2 + 3 = 5 is wrong, but 2 = 3 + 5 is okay.

b. The child indicates that 5  3 + 2 is not okay, writes an equals sign between the 5 and the 3, but offers no rationale for writing the equals sign other than, "It's supposed to be there."

c. The child says that 6 = 4 + 2 is wrong; changes it to 4 + 2 = 6; explains, "You don't put equals first" but offers no other explanation for the equals sign following the addition sign.

Category 2. The child's behaviors suggest that the child views the equals sign as a one-directional operator which separates a problem and its answer. Behaviors which typify this view of equality are as follows:

a. The child indicates that 6 = 4 + 2 is wrong, changes it to 4 + 2 = 6, and explains that the answer comes after the problem.

b. The child rejects 4 + 2 = 6, accepts 6 = 4 + 2 because the sum comes first.

c. The child indicates that 3 + 2 = 2 + 3 is wrong, changes it to 3 + 2 = 5 / 2 + 3 = 5.

d. The child rejects 3 + 2 = 2 + 3, changes it to 3 + 2 + 2 + 3 = 10.

e. The child reads 6 = 2 + 4 as "Two plus four equals 6."

f. The child indicates that 3 = 3 is wrong, changes it to either 3 + 0 = 3 or 3 = 3 + 0, depending on the child's placement of the sum on the right or the left side of the equals sign.

g. The child says that 3 = 3 is wrong, changes it to either 3 + 3 = 6 or 6 = 3 + 3.

h. The child accepts 3 + 2 = 5 + 1 because 3 + 2 equals 5, but rejects 1 + 5 = 3 + 2 because 1 + 5 $\neq$ 3.

i. The child rejects 3 + 2 = 4 + 1 because 3 + 2 $\neq$ 4.

Category 3. The child's behaviors suggest that the child views the equals sign as an operator separating a problem from the answer, but does not insist that either the problem or the answer

must come first. Typical behaviors clustered in this category are as follows:

a. The child accepts both $3 + 2 = 5$ and $5 = 3 + 2$ as being correct.

b. The child accepts both $3 + 2 = 5 + 1$ and $1 + 5 = 3 + 2$ as being okay, because in one case $3 + 2 = 5$ is correct, and in the other case $5 = 3 + 2$ is correct.

c. The child rejects $3 + 2 = 4 + 1$ because neither $3 + 2 = 4$ nor $2 = 4 + 1$ is correct.

d. The child rejects $3 + 2 = 2 + 3$ because neither $3 + 2 = 2$ nor $2 = 3 + 2$ is correct.

e. The child indicates that $3 = 3$ is not right, and says that both $3 + 0 = 3$ and $3 = 3 + 0$ would be right.

Category 4. The child's behaviors suggest that the child interprets the equals sign as symbols which denote that two expressions are names for the same number. Behaviors representative of those in this category are as follows:

a. The child accepts $3 = 2 + 1$ and $2 + 1 = 3$ explaining that both $2 + 1$ and 3 are the same number.

b. The child says $3 = 3$ is okay because both are the same number.

c. The child, in accepting $3 + 2 = 4 + 1$, reasons: $3 + 2$ is 5, $4 + 1$ is 5, both are equal to 5, so they are equal.

The distinctions among the five views of equality can best be observed in the nature of students' responses to a common stimulus. For example, consider the following (Table 12) responses to the question, "Is $2 + 3 = 4 + 1$ all right?" Responses cited in Table 12 on the following page are typical of those one would expect from a student with a particular conception of equality, as defined in Categories 0-4.

## Analysis of Data

Using the above descriptions of children's interpretations of equality as a guide, analyses were made of the students' performances on tasks included in the first (December) and the final (June) external evaluations. On each test the children's behaviors were not consistent for each task, but on each test a child's behavior patterns generally clustered in one of the above categories. Thus on each evaluation it was possible to identify a child's view of equality by one of the five category descriptors. A summary of these analyses is reported in Table 13 on page 35.

Table 12
Typical Responses to the Question: Is 2 + 3 = 4 + 1 all right?

| Category Level | Illustrative Response | Explanation of Response |
|---|---|---|
| 0 | It is not right. The one should be a five. | The student's attention is focused on an order relation, ignoring the addition and equals signs. |
| 1 | It is not right. Changes it to 2 + 3 + 4 = 1 or 2 = 3 + 4 + 1, depending on whether, in the student's view, the equals sign comes immediately after the first number or before the last number. | In this case the student insists that the equals sign be placed in a specific location within a sentence, but assigns no meaning to the sign. |
| 2 | It is not right. Explains that the sum of 2 and 3 is not 4, or that 3 is not the sum of 4 and 1, depending on whether the direction of the operator is from left to right or from right to left. | The student identifies two numbers as constituting an addition problem, a third number as the sum, and then makes a judgment as to whether or not the third number is the sum of the first two. The equals sign helps the students decide which numbers make the problem and which number could be the sum. |
| 3 | It is not right. Explains that neither 2 + 3 = 4 nor 3 = 4 + 1 are right. | The student identifies two possible problems. In one case the sum is on the left of the equals sign and in the other case the sum is on the right. The equals sign is only a symbol which helps the student identify a problem and a sum, but in this case the student recognizes the existence of two problems. |
| 4 | It is all right. Explains that both 2 + 3 and 4 + 1 are names for 5. | In this case the student views the equals sign as a symbol which denotes a relation between two names for the same number. |

It can be observed from the data presented in Table 13 that at the time of the first evaluation (December), four of the seven experimental students viewed equality in a manner as described by either Category 3 or 4, and that

Table 13
Students' Views of Equality by Categories

| Pair | Experimental | | Control | |
|---|---|---|---|---|
| | December | June | December | June |
| 1 | 4[a] | 4 | 2 | 2 |
| 2[b] | 3 | - | 3 | - |
| 3 | 3 | 2 | 3 | 4 |
| 4 | 1 | 3 | 2 | 3 |
| 5 | 3 | 3 | 0 | 0 |
| 6 | 1 | 1 | 0 | 0 |
| 7 | 1 | 2 | 1 | 0 |

[a]Numbers correspond to category labels described above.

[b]Students in pair 2 did not participate in the final evalution.

only one student in the control group viewed equality as described in Category 3. At the end of the experimental period (June), three of the six students in the experimental group had a concept of equality corresponding to Categories 3 or 4, and only one experimental student had not progressed beyond Category 1. Although two students in the control group viewed equality as described by either Category 3 or 4, it can be observed that three of the six control students had no comprehension of the meaning of the equals sign. Thus, the data in Table 13 indicate that the students in the experimental group generally had a deeper understanding of equality at the time of the first evaluation and that they maintained this advantage throughout the year. It can also be observed that the growth in the students' conceptualization of equality was minimal.

The data obtained from this teaching experiment do not support the conjecture that if students (first graders) are provided with appropriate instructional experiences in which they encounter the use of the equals sign in a variety of sentence forms, they will learn to view equals as a relation between two numbers. In fact the evidence is quite to the contrary. Namely, the development of a first grade student's concept of equality as a relation is not merely a matter of instruction. In particular, the data suggest that a first grade student's experiences with equality in the context of addition, subtraction, and missing addend equations are not in themselves totally sufficient to develop the student's conceptualization of equality as a relation between two names for the same number. Rather, among first grade students the development of this concept of equality appears to be dependent on other factors such as the student's ability or intellectual development.

## IMPLICATIONS FOR CURRICULUM DEVELOPMENT AND FURTHER RESEARCH

The results from this study of first graders' conceptualizations of equality suggest that curriculum designers and mathematics researchers should give more thoughtful consideration to the development of equality as a concept. This is especially true if the ultimate goal is for students to view equality as a relation between two numbers. In particular, attention should be given to the development of a sequence of instructional activities appropriate for specific grade levels, explicitly directed toward the development of equality as a relation and to the appropriate placement of these activities within the mathematics curriculum.

Specifically, there are four problems which should be addressed.

Sentence Structure. In most curricula students primarily encounter equations of the form $a \circ b = c$. The results from this study indicate that experiences with other sentence structures ($a = a$, $a = b \circ c$, and $a \circ b = c \circ d$) had no adverse effects on the students' attainment of addition, subtraction and place value concepts and skills. Furthermore, such experiences contributed to the students' development of a more flexible view of equality. However, the data from this study did not indicate that that experiences with various sentence structures contributed to the first graders' understanding of equality as a relation.

It is conjectured that, if the intensive instruction on equality were continued in later grades, the effects on the students' learning to view equality as a relation may be significant. This is an issue which needs to be investigated further.

Symbolism. The results from this study and other studies indicate that first grade students view the equals sign as an operator. That is, as a symbol which separates a problem and its answer. Data from other studies indicate that this view prevails for several years. Thus, if students are to eventually learn to view the equals sign as a symbol which connects two names for the same number, they must unlearn a point of view which has become firmly entrenched.

Perhaps curriculum designers should consider the use of a symbol such as an arrow ($\longrightarrow$) to denote the one-way operation until such time as the students are capable of understanding equality as a relation.

Placement. Data from this study and other studies clearly indicate that students do not detect a view of equality as a relation from their work with equality in the context of computational problems. The data suggest that students need instructional activities which explicitly develop the concept of equality as a relation. Therefore, a careful study should be undertaken to determine at which grade (age) level most students are capable of understanding equality as a relation and then investigate the nature of instructional activities which would be most effective in teaching this concept.

Language. Data from this study indicate that some students (control group) who were taught to read the equals sign (=) as "the same as" developed

the same flexibility in accepting a variety of sentence structures as the students who received the experimental treatment. This suggests that to teach students to read the symbol (=) as "equals" may inhibit the students' development of a more flexible notion of equality. This suggests that the effects of using different verbal names for (=) should be investigated in studies which are explicitly concerned with this issue.

# APPENDIX A

## SUMMARY OF CONCEPTS AND SKILLS

## AND

## SUMMARY OF STUDENT BEHAVIORS

Introduction of Equality

A. Concepts and skills taught in this unit included:

1. Concept of weight.

2. Skill in using a balance to weigh an object.

3. Concept of a unit weight.

4. Concept of two objects having the same weight.

5. Concept of symmetry in weighing objects--that is, an object will weigh the same whether it is placed on the right or left side of the balance.

6. Skill of recording the weight of an object on a worksheet.

7. Concept of two sets of units being equal in weight.

8. Skill of constructing a balance model based on information provided in a written exercise.

9. Skill of using equals and not equals sign to show whether two sets of units do or do not have the same weight.

10. Concept of truth value of a statement.

B. Observations of students' behaviors included:

1. All students had an intuitive understanding of: (a) These (objects) weigh the same, and (b) This (object) is heavier. That is, by holding objects in their hands the students could determine whether two objects weighed the same or which one was heavier.

2. The students knew that a scale is used to weigh objects.

3. The students had no difficulty in understanding how a balance is used to weigh objects. That is, the object to be weighed is placed on side; units are placed on the other side to make the pans level; and the weight of the object is determined by counting the units.

4. The students had no difficulty in accepting that two different objects could weigh the same.

5. The students knew that moving an object from one side of the balance to the other side would not change the weight of the object.

6. The students understood the procedure of recording the weight of an object, but most (all but 2) experienced considerable frustration in relating the left-right sides of the balance to the left-right

sides of a worksheet. Basically, these students did not know the meaning of right and left. Labeling the sides of the balance and the worksheet as L and R made it easier for these children to establish a correct correspondence between the balance and worksheet.

7. Students accepted that two sets of units could have the same weight. In determining whether or not two sets of units had the same weight, the students usually relied on the number property of the sets rather than on whether or not the pans were level. That is, they knew that if the number of units in each set was the same, the weights would be the same, and conversely.

8. "Equals" means same amount on both sides--pans are level. Students quickly learned to use an equals sign to show that weights were the same; a not equals sign to show that weights were unequal.

9. Students had no difficulty in learning to construct a balance model based on information provided by written instructions (e.g., $\boxed{2} \bigcirc \boxed{4}$) and then using = or ≠ to complete a sentence (e.g., $\boxed{2} \neq \boxed{4}$).

10. Students learned to read orally sentences like 5 = 5 and 5 ≠ 4 without a great deal of difficulty.

11. After working 2 or 3 exercises, three students (N, M, J) did not consistently use a balance in completing sentences like $\boxed{3} \bigcirc \boxed{3}$ or $\boxed{3} \bigcirc \boxed{2}$. The other students continued to use a balance, but their behavior indicated that they did so because (a) they felt that they were expected to use a balance or (b) they wanted to verify their answer.

12. Students had no problems in stating that 5 = 5 was right, but at first they did not accept the fact that one could write 4 = 6. Later, however, they learned to indicate by making an X next to the equation, that statements like this were not correct.

Introduction to Addition (Sums ≤ 6)

A. Concepts and skills taught in this unit included:

1. Concept of addition--using two distinct sets to weigh an object.

2. Skill in recording weights by a pair of numbers.

3. Skill in solving equations of the form $a = \square + \square$.

4. Skill in reading addition equations of the form $a = b + c$.

5. Skill in constructing balance models for equations of the form $a = b + c$.

6. Skill in solving equations of the form $\square = a + b$.

7. Skill in using = or ≠ to complete equations of the form $a \bigcirc b + c$.

8. Concept of truth value of statements of the form $a = b + c$.

B. Observations of students' behaviors included:

1. Students had no problems in constructing two sets to determine the weight of an object.

2. Except for two students (J, M), the students at first had considerable difficulty in recording a weight as a pair of numbers (Form $\square$ $\square$). The students tended to record the total weight (sum) in one box.

3. Students readily accepted zero as an addend.

4. Students accepted the use of the plus sign (+) to mean "How many altogether."

5. Except for three students (D, J, M), at first students had problems reading addition equations. Common errors were (1) omitting equals and/or plus; (2) reading equals as plus and vice versa.

6. Given a set of n units on the left side, at first the students would put n units in one region on the right side and then write $n = n + 0$ or $n = 0 + n$.

7. Students readily accepted equations of the form $n = 0 + n$ and $n = n + 0$, because the numbers were the same on both sides. They ignored the zero. Consequently the students were reluctant to accept equations like $2 = 1 + 1$, because the numbers were different.

8. Except for two students (Da, A), the students quickly learned to construct a balance model to show an addition equation, $a = b + c$.

9. In the process of completing exercises of the form $\square = \square + \square$, two students turned the worksheets around so the equations were of the form $\square + \square = \square$. In either case, students first constructed the addend sets and then the sum set. At first students constructed the sum set by trial and error, but then discovered that they could count the addend sets to determine the sum set.

10. Students had no problems in learning to use the balance to solve problems of the form $\square = a + b$.

11. Students could tell whether equations of the form $a = b + c$ or $a \neq b + c$ were right, but most based their answers on the number of units on each side rather than on whether or not the pans were level.

12. After showing a sentence like $5 \neq 1 + 3$ on the balance, students could change the balance to make it equal.

13. Students quickly observed that a model showing $5 = 4 + 1$ could be changed to show $5 = 1 + 4$ by switching the addend sets.

14. Except for two students (D, M), the students were generally unable to find all solutions for equations of the form $a = \square + \square$. Except for using the CPA (comment 13 above), all work was by trial and error.

Introduction to Missing Addends: (Sum $\leq 6$)

A. Concepts and skills taught in this unit included:

1. Concept of a missing addend--relationship between balance models and equations of the forms $a = b + \square$ and $a = \square + b$.

2. Skill in solving missing addend problems ($a = b + \square$ and $a = \square + b$), grouped by sum families.

3. Skill in reading missing addend equations.

4. Concept of relating addition problems and missing addend problems. For example, establishing a relationship between $6 = 2 + \square$ and $\square = 2 + 4$.

5. Skill in solving addition and missing addend problems which contained 3 addends ( $\square = a + b + c$, $a = \square + \square + \square$, $a = b + c + \square$, etc.).

B. Observations of students' behaviors included:

1. Students (except N) had no problems in interpreting the meaning of missing addend problems, constructing the appropriate balance models, and recording the answer.

2. Except for two students (D,M) students at first solved missing addend problems, grouped by sum families, as independent problems, That is, they removed all units from the balance. Later, the students observed (a) that they could leave the sum set unchanged, (b) that after solving $5 = \square + 3$ they could solve $5 = 2 + \square$ without constructing a new model, and (c) a model for $5 = 3 + \square$ could be obtained by switching the addend sets from one region to the other region.

3. Students did not rely on a strict one-to-one correspondence between the balance model and an equation to solve an equation. That is having set up the balance to show $4 = 3 + 1$, they could solve $\square = 3 + 1$, $\square = 1 + 3$, $4 = 3 + \square$, $4 = 1 + \square$ and $4 = \square + 3$.

4. Although students observed relationships between certain pairs of

missing addend problems ($4 = 3 + \square$ and $4 = \square + 1$) and pairs of addition problems ($\square = 4 + 1$ and $\square = 1 + 4$), the students seemingly did not relate missing addend and addition problems. For example, after solving $5 = 4 + \square$, they would remove the units from the balance before solving $\square = 4 + 1$.

5. Two students (D, M) discovered a pattern in using the balance to solve missing addend problems in a sum family. For example, after solving $5 = 3 + \square$, they would move one unit from the set with 3 units to the set with 2 units to solve $5 = 2 + \square$.

6. Students had no difficulty in solving problems with 3 addends.

Introduction to Subtraction (minuend $\leq 6$)

A. Concepts and skills taught in this unit included:

1. Concept of subtraction (take away).

2. Skill in writing equations to show a take away situation ($a = b - c$).

3. Skill in constructing a balance model to show a subtraction equation.

4. Skill in reading subtraction equations.

5. Skill in solving subtraction problems ($\square = a - b$).

6. Skill in using $=$ and $\neq$ to complete sentences of the form $a \bigcirc b - c$.

7. Skill in solving problems of the forms $a = b - \square$ and $a = \square - \square$.

8. Concept of the truth value of a subtraction equation.

B. Observations of the students' behaviors included:

1. Students had no problems in solving the introductory subtraction (take away) problems; for example, putting 3 units on left side, 5 units on right side, and then removing 2 units from right side to make the balance level.

2. Students generally had considerable difficulty in learning to describe (verbalize) the subtraction process, that is, to write subtraction equations, to read subtraction equations, to construct a balance model to show a subtraction equation. Their difficulty seemed to be attributed to the following facts: (a) There was not a clear correspondence between the balance model and the equation, (b) The subtraction process was more involved than the addition process, and (c) Subtraction problems were treated as addition problems.

3. From the beginning one student (D) did not use a balance to solve subtraction problems.

4. Except for two students (D, M) students had difficulty in distinguishing between $\square = a - 0$ and $\square = a - a$.

5. Students had no problems in using = or ≠ to complete sentences of the form a ○ b - c.

6. Students readily understood equations of the form $a = b - \square$. Their only difficulty in solving these problems was that they would place the units they took away, one at a time, on the table with the other units; therefore, they had no easy way to determine how many they had taken away.

7. Students learned, without difficulty, to solve equations of the form $a = \square - \square$.

8. Three students (D, J, M) observed a pattern in solving equations in exercise sets where each equation was of the same form ($a = \square - \square$). The other students worked such problems by trial and error, often repeating a specific equation several times.

9. Two students (D, M) were not consistent i. reading subtraction problems from left to right. They would on occasion read an equation like 3 = 5 - 2 as "Five take away two equals three."

10. By the time this unit on subtraction was completed, three students (D, J, M) were solving subtraction problems without using their balance. However, two of these students (J, M) would frequently use their balance to verify their answers.

11. Students had no problems in determining whether or not a subtraction equation was correct. Also, they could change an equation to make it correct, with or without using a balance.

12. One student (D) consistently reads = as "the same as." He explained to the other students that "equals" and "the same as" mean the same thing, but the other students continued to read = as "equals."

Introduction to Greater Than and Less Than ($N \leq 10$)

A. Concepts and skills taught in this unit included:

1. Concepts of greater than and less than.

2. Skill in showing greater (less) than relations on a balance.

3. Skill in reading sentences which include the phrases "is greater than" and "is less than."

4. Skill in identifying numbers greater (less) than a given number.

5. Skill in using > or < to complete sentences of the form a ○ b.

6. Skill in reading sentences of the form a > b or a < b.

7. Concepts of one more (less) than.

8. Skill in identifying a number which is one more (less) than a given number.

B. Observations of students' behaviors included:

1. Prior to the introduction of this unit the students had demonstrated an understanding of "more than" and "fewer than." Except for two students (Da, A), the students related this knowledge to the terms "greater than" and "less than."

2. Except for two students (Da, A), the students could verbalize how greater than and less than relations could be detected from the balance. For example, some said, "The side with more goes down." Also, some reasoned, "9 is picking up 7, so 7 is less."

3. Except for two students (Da, A), the students readily learned to read sentences such as: ☐ is greater than 7; 6 is less than ☐ ; 9 is greater than 6.

4. Students completed most exercises in this unit without using a balance. They relied instead on their knowledge of the ordering of numbers.

5. Students made numerous errors as they completed the written exercises involving the terms "is greater than" and "is less than." The errors were not due to conceptual difficulties, rather the errors were attributed to the students' failure to read the sentence. For example, in completing a sentence like "7 is less than ☐ " they focused their attention on 7 and less, and then wrote a number less than 7 in the box. When asked to read the completed sentence, the students immediately detected that it was wrong and corrected the error.

6. The students experienced considerable difficulty in learning the signs > and < . They could not remember which sign was greater than and which was less than. Clues like "The arrow points to the pan which is up" did not help the students in deciding which sign to place on the balance.

7. The students (except Da, A) could complete exercises like a ◯ b if the following clue was written on the board: less . . . < ; greater . . . > .

8. The students were not successful in learning a technique for using a balance to obtain an answer for questions like " ☐ is one more than 7" or " ☐ is one less than 9." Rather, they avoided the use of a balance in answering such questions and relied entirely on their knowledge of the ordering of numbers.

9. The concepts of "is one more than" and "is one less than" were very difficult for all but 2 students (D, M). The assessment is based on the fact that the students were generally unsuccessful in completing sentences like " ☐ is one more than 8."

10. The students were given money exercises involving how many more. For example, a student was given 5¢; an object priced at 7¢ was held up; the student was asked if he had enough to buy the object; then how many more pennies did he need. Students could use the balance to solve these problems, but they were more proficient in using their fingers or mental computations to find the answer.

Introduction to Place Value ($N \leq 19$)

A. Concepts and skills taught in this unit included:

1. Skill in rote counting to 19.

2. Skill in counting/constructing sets with up to 19 units.

3. Skill in reading/writing numerals 11-19.

4. Concept of a long (ten) being equivalent to ten units.

5. Concept of a number greater than 10 being so many more than 10--13 is 3 more than 10.

6. Skill in using longs (tens) and units to represent a number greater than 10.

7. Skill in representing a number greater than 10 in expanded form.

8. Concept of truth value of statements involving expanded form.

B. Observations of students' behaviors included:

1. Students readily learned rote and rational counting skills.

2. Students readily learned to read and write numerals 11-19.

3. Students readily accepted 1 ten as being equivalent to 10 units.

4. Students readily learned to answer questions like "13 is how many more than 10?" and to use the balance to answer the questions.

5. Students exhibited no difficulty in completing expanded forms for exercises like: 18 = ☐ + ☐ ; ☐ = 1 ten + 5; ☐ = 10 + 3. Except for two students (D, M) the students at first used only units in constructing sets for numbers greater than or equal to 10, except that they would use a long to represent 1 ten. After some experience with exercises involving 1 ten and 10, they began to use longs in constructing sets for numbers greater than or equal to 10. At first the students made many errors in completing exercises like

18 = □ + □. They wrote: 18 = 1 + 8. This apparently was not a conceptual problem, rather they intended for the "1" to mean "1 ten." The error became less frequent as the students accepted "1 ten" and "10" as being equivalent.

6. Except for one student (D), students used their balances in determining the truth value of statements like: 14 = 10 + 4 or 12 = 1 ten + 2. One student "verified" that 15 = 5 tens + 1 by putting 15 units on one side and 5 units and 1 long on the other. When asked to read the statement she corrected her answer.

7. Except for one student (D), the students do not seem to appreciate the place value aspect of numerals for numbers greater than 10. (They had not received explicit instruction on this concept.) For example, in using the balance to solve 14 = □ + □ , they could show 14 on the left side as 14 units or 1 long and 4 units, immediately put a long on the right side (they know that $14 > 10$); write 10 in first box, add units one at a time to the right side until it balanced, and then count the units on the right side before writing 4 in the second box.

Addition, Subtraction, Missing Addends: $7 \leq$ Sum/Minuend $\leq 10$

A. Concepts and skills taught in this unit included:

1. Concept of sum families--finding all solutions to equations of the form a = □ + □ .

2. Skill in solving addition problems, □ = a + b.

3. Skill in solving addition problems with 3 addends.

4. Skill in using = and ≠ to complete sentences of the form a ◯ b + c or a ◯ b + c + d.

5. Concept of the truth value of an addition equation.

6. Skill in solving missing addend problems of the form a = b + □ , a = □ + b, a = b + c + □ , a = b + □ + c or a = □ b + c.

7. Skill in solving subtraction equations of the form □ = a - b or a = b - □ .

8. Skill in using = or ≠ to complete sentences of the form a ◯ b - c.

9. Concept of the truth value of a subtraction equation.

B. Observations of the students' behaviors included:

1. Except for two students (Da, A), the students had a good command of the addition facts with sums 6 or less. That is, these five students had memorized most of the easy addition facts. The two students who had not learned the easy facts received additional drill on the easy

facts while learning the harder facts.

2. Students had no difficulty in extending the computation of addition facts to sums through 10. Three students (D, J, M) computed the harder facts without the aid of a balance. They used their fingers, but they could demonstrate the computation process on the balance when asked to do so.

3. Two students (D, M) could make up addition problems with sums greater than 10.

4. Students had no difficulty with 3 addend addition problems.

5. Students demonstrated that they could use = and ≠ to complete sentences of the form a ◯ b + c. Except for 2 students (D, M) the students used their balances. Interestingly, most students counted the units on each side of the balance to determine whether to use = or ≠, rather than relying on the position of the pans.

6. Students could determine the truth value of an addition equation. Given an equation and an inappropriate balance model, the students would not say that the model was wrong. Rather they would describe the problem shown on the balance then tell how it should be changed to make it show the written equation.

7. Students had no difficulty with missing addend problems. Except for two students (D, M), the students used their balances.

8. In solving missing addend problems, grouped by sum families, the students would leave the sum set on the balance and construct only the addend sets.

9. Students had no difficulty in solving the harder subtraction problems.

10. Most incorrect responses were due to counting errors rather than misconceptions.

11. Occasionally a student would reverse the pans on his balance, that is, put the 2-region pan on the left side. This change, however, did not cause the students problems in solving equations.

12. At the end of the units students were beginning to discover more efficient ways of constructing balance problems by modifying an existing model to solve the next problem. For example, having solved 8 = 5 + 3 on the balance, the student would remove one unit from the 8 units and add one unit to the 5 units to solve 7 = 6 + ☐ . In other cases they would modify the addend sets on the balance to solve a different addition problem. For example, after solving ☐ = 5 + 4 rather than removing the addend sets before solving ☐ = 4 + 3, they would simply remove one unit from each addend and then determine the sum.

13. Two students (D, M) when asked to make-up problems occasionally wrote them in the form a + b = c or a - b = c.

14. In solving addition and subtraction problems on a balance, most students use only one side of the balance. For example, they only count the units remaining after the subtrahend had been taken away to determine the difference in subtraction problems. Similarly, they count the addend sets together without constructing a separate sum set.

15. Students were introduced to buying/selling problems. At first, the students did not relate these problems to addition and subtraction. For example, when shown an object which cost 2¢ and another object which cost 3¢, the students would not accept that the total cost could be determined by adding 2 and 3.

16. Students have no problems with zero as an addend, but most are confused by zeros in subtraction. Specifically, they confuse problems of the form $\square = a - a$ with problems of the form $\square = a - 0$ when demonstrating the solutions on a balance. The problem seems to be that they first mentally determine the answer then took away the answer. To solve $\square = 9 - 9$, they put 9 units on the right side, think 9 take away 9 is zero, take away zero, thus the answer is 9. Similarly, the answer to $\square = 9 - 0$.

17. Except for two students (D, M), the students had considerable difficulty in making up problems on their own, even when it was suggested that they first show a problem on their balance. It seemed that they were hesitant to make a decision as to which numbers to use in stating the problem.

18. By the time this unit was completed, all students were attempting to work problems without the aid of a balance. However, when they encountered a problem they could not solve mentally or on their fingers, they returned to the balance. In other instances, a student would complete an entire exercise set without using a balance, usually correctly, and then use the balance to verify the answers.

19. At the end of unit, when sums of 9 and 10 were introduced, the students were completing the third cycle of being presented addition, missing addend, and subtraction problems. This spiraling procedure seemed to have an adverse affect on three students (A, Da, N) in that these students occasionally misread problems and/or construct a balance model appropriate for another type of problem. Prior to the third cycle these students had exhibited no evidence of confusing the three types of problems.

Introduction to Conventional Forms of Writing Problems

A. Concepts and skills taught in this unit included:

1. Skill in solving addition problems expressed in the form $a + b = \square$ or $\begin{array}{r} a \\ +\ b \\ \hline \end{array}$.

2. Skill in solving subtraction problems expressed in the form

$a - b =$ or $\begin{array}{r} a \\ -b \\ \hline \end{array}$.

3. Skill in solving missing addend problems expressed in the form $a + \square = b$, $\square + a = b$, $\begin{array}{r} a \\ +\square \\ \hline b \end{array}$

B. Observations of students' behaviors included:

Students were not introduced to the conventional forms until mid-April. Their first encounter with such problems occurred when their assignments included pages from a textbook. None of the students experienced any difficulty in interpreting the meaning of the problems or in reading the problems.

Addition Facts (Sum $\leq$ 13)

A. Concepts and skills taught in this unit included:

1. Skill in using a balance to find several pairs of addends which have the same sum--Find more than one solution for equations of the form: $a = \square + \square$.

2. Skill in computing addition facts, sum greater than 10.

3. Concept of relationships which establish the equivalence of two addition problems--Solve an equation of the form $a + b = \square + c$ without computing sums.

4. Skill in using $=$, $>$, and $<$ to complete sentences of the forms: $a \bigcirc b + c$ and $a + b \bigcirc c + d$.

5. Concept of the truth value of an addition equation.

B. Observations of the students' behaviors included:

1. The students had no difficulty in finding at least three pairs of addends for each sum: 11, 12, and 13. In completing the exercises the students consistently made use of the commutative property. For example, having written $12 = 7 + 5$, they immediately wrote $12 = 5 + 7$ as the next equation. One student (D) used a sequence pattern in completing the exercise sets. For example, he wrote $13 = 10 + 3$, $13 = 9 + 4$, $13 = 8 + 5$, etc.

2. The students had no difficulty in using a balance to solve problems of the form $\square = a + b$. Having constructed the addend sets on the right side of the balance, they would put a ten (long) on the left side and add units to make it balance. To determine the sum they counted on from 10. When the sum was less than 10, the students immediately removed the long and replaced it with units. After some experience in using a balance, some students abandoned the balance and began using their fingers as the latter method was quicker;

others counted the addend sets rather than constructing a sum set.

3. By the time the unit was completed, only one student (A) consistently used a balance to determine sums. One student (D) used his fingers or recalled sums from memory. The other students had developed a new strategy; they used units to construct addend sets but placed them on the table rather than on the balance.

4. The students were reviewed each day on basic facts with sums 10 or less. Some exercises included missing addend problems. Since the students had not encountered problems of this type for several weeks, they tended to treat such exercises as addition problems.

5. Students learned to verbalize a strategy for modifying a balance model used to solve one problem to obtain a model for a second problem. For example, the balance model shown 13 = 8 + 5. To solve $\square$ = 7 + 6, they say move one unit from the set of 8 units to the set of 5 units. They knew the sum was 13 without recounting the sum set. Continuing to solve $\square$ = 7 + 5, they would say take one unit from the set of 5 units and take one unit from the left side (sum). They would determine the new sum without counting the sum set. The above behaviors were observed during group discussions. However, when completing written exercises, only two students (D, J) used these relationships. The others would complete such exercises as an independent problem.

6. Students experienced considerable difficulty with equations which had two addends on each side of the equals sign. They could not at first construct appropriate balance models; they ignored the addend on the side which contained a box ( $\square$ ) and wrote the sum of the addends on the other side in the box, e.g., 8 + 5 = 4 + $\boxed{13}$ ; or they wrote the sum of all these addends in the box, e.g., 8 + 5 = 4 + $\boxed{17}$ . Eventually the students learned to construct balance models and correctly complete the equations, but in doing so they did not use a relationship between problems, even problems like 5 + 4 = $\square$ + 5. Except for one student (D), each problem was solved by trial and error; that is, they found the missing addend by placing units on the balance one at a time until it balanced, then they counted the units. The one student who did not use the trial and error strategy used a computation strategy. For example, to solve 5 + 8 = 6 + $\square$ , he added 5 and 8, then asked himself, "What do I add to 6 to get 13?"

7. When the children worked with problems of the form similar to a + b = c + $\square$ , the balances were set up so that there were 2-region pans on each side of the balance. During review exercises the students were given problems of the form $\square$ = a + b and a = b + $\square$ . The students made many errors on problems of this type until one of the 2-region pans was replaced by a 1-region pan. Seemingly, the students did not--perhaps they could not--establish on their own initiative a relationship between the balance model and the written equation.

In particular, the seemingly related each pan to a side of the equation and did not relate the fulcrum (a wooden block on the bal-

ance) to the equals sign in the equation. This was evidenced by the haphazard manner in which they constructed or interpreted balance models. For example, for solving $\square = 7 + 6$ one student put 7 units on the left side, 6 units on the right side, another unit on the right side, and wrote $1 = 7 + 6$. Another student in solving $13 = 10 + \square$ set up the balance to show $13 = 10 + 3$, but wrote $13 = 10 + 13$. Her reason was that it takes thirteen to make it balance. As another example, a student solved $7 = 6 + \square$ by putting sets of 7 units and 6 units on the right side and writing $7 = 6 + 13$. Also, one student solved $8 + 5 = \square$ by putting 8 units on the left side, 5 units on the right side, and wrote $8 + 5 = 3$.

This confusion in relating a balance model to an equation is a probable cause for the difficulties the students had in solving missing addend problems. It should be noted that the students were not explicitly instructed on how to relate a balance model to an equation. Inferences to the relationship was made in exercises of the following types: (a) The students were asked to read an equation and then asked, "What does the sentence tell us to do?" (b) The students were shown a balance model and an equation and asked, "Does the equation tell us what is shown on the balance?" A possible reason for the confusion about relating a balance model to an equation was the introduction of conventional writing forms towards the end of the year without explicit instruction in how to relate an equation like $3 + 4 = \square$ to a balance model.

8. The students could detect more than and less than relations from a balance model, and they could verbalize a rationale for the relation, but they had considerable difficulty in completing sentences of the form $a \bigcirc b + c$ or $a + b \bigcirc c + d$. The reason for this difficulty was that they did not learn the signs $<$ and $>$. To avoid this difficulty they would use the $\neq$ sign, e.g. $3 + 8 \neq 7 + 5$ rather than writing $3 + 8 < 7 + 5$. Their rationale for determining the appropriate relationship was based on the number of units on each side of the balance, and not on the position of the pans. Thus in completing exercises like $9 + 4 \bigcirc 6 + 6$ they would compute with or without a balance both sums, reason, "13 is not equal to 12," and write $9 + 4 \neq 6 + 6$.

9. Students had no difficulty in determining the truth value of statements like $3 + 8 = 7 + 4$ or $3 + 8 = 7 + 5$. To do so, they computed each sum and then compared the sums. However, except for one student (D), the students generally indicated that all sentences of the form $a + b < c + d$, $a + b > c + d$, $a < b + c$, $a > b + c$ were wrong. Their rationale was that this was not equal so it must be wrong. Likewise, $3 + 9 \neq 10 + 1$ was wrong because it was not equal. It appeared that the students associated equals with being right and not equal with being wrong, except that equations like $13 = 7 + 3$ were wrong because the sum of 7 and 3 was not 13.

Place Value ($N \leq 29$)

A. Concepts and skills taught in this unit included:

1. Skill in rote/rational counting.

2. Skills in reading/writing numerals.

3. Skill in using tens (longs) to construct sets for the numbers 20-29.

4. Skill in determining the number of a set containing 2 tens (longs) and up to 9 units.

5. Skill in expressing the numbers 20-29 in expanded form.

B. Observations of students' behaviors included:

1. Students readily learned rote counting, rational counting, reading and writing skills.

2. Only two students (A, Da) had difficulty in learning to count on from 20.

3. Two students (A, Da) reverted to using only units in constructing sets, rather than using a combination of tens and ones.

4. Only one student (D) is able to complete exercises like 24 = ☐ + ☐, ☐ = 20 + 4 or ☐ = 2 tens + 4 without constructing a balance model.

5. All students readily accepted 20 and 2 tens as being equivalent.

Addition with at Least One 2-digit Addend: No Regrouping

A. Concepts and skills taught in this unit included:

1. Skill in adding 2-digit number and a 1-digit number.

2. Skill in adding two 2-digit numbers.

B. Observations of students' behaviors included:

1. All students were able to use a balance to find at least 4 solutions to an equation like 26 = ☐ + ☐. Most students made use of the commutative property in determining the solutions.

2. Students readily learned to use a balance to solve problems like ☐ = 15 + 4 and $\begin{array}{r} 23 \\ +\ 2 \\ \hline \end{array}$. Only two students (A, Da) apparently did not detect the addition algorithm--add ones then tens. Two students (A, Da) worked very slowly and made many errors because they avoided using tens (longs) in constructing sets. They have considerable difficulty in counting on from 10 or 20.

3. Except for two students (A, Da), the students learned to use a balance to add two 2-digit numbers. These students first counted the tens and counted on to find the sum. Only one student (D) detected the addition algorithm and worked the problems without constructing a model.

Subtraction with 2-digit Minuends: No Regrouping

A. Concepts and skills taught in this unit included:

1. Skill in subtracting a 1-digit number from a 2-digit number.

2. Skill in subtracting a 2-digit number from a 2-digit number.

B. Observations of students' behaviors included:

Only four students (D, E, J, N) worked on this unit. Three students (E, J, N) learned to complete the exercises by constructing an appropriate balance model. They used combinations of tens and ones to construct sets, and took away the subtrahend set by removing tens and ones. The other student (D) quickly detected the subtraction algorithm and solved the problem without the aid of a balance.

# APPENDIX B

## SAMPLE OF INSTRUCTIONAL MATERIALS: EXPERIMENTAL TREATMENT

Representative lessons, teacher guides and student worksheets from the experimental treatment are included in this appendix. The content of these lessons reflects the development of the following major topics: equality, addition, missing addends, subtraction, place value, and order. This selection of lessons does not include drill exercises or repetitions within major topics. Rather, the selection of lessons only highlights the important features of the experimental treatment.

Introduction: Concept of Weight

Purpose(s): a) To introduce the concepts that some objects weigh the same and that some objects are heavier than others.

b) To motivate the introduction of the balance.

Materials: Ten objects, each with a weight of 1,2,3,4,5, or 6 units.

Vocabulary: Which one is heavier?
Which one weighs more?
Do they weigh the same?

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Hold up two objects - different sizes; weights of 2 and 6..... | DO THESE WEIGH THE SAME?<br>WHICH ONE IS HEAVIER? |
| Give objects to a child..... | DO THEY WEIGHT THE SAME?<br>WHICH ONE IS HEAVIER?<br>HOW CAN YOU TELL? |
| 2. Hold up two objects - different sizes; same weight (3)..... | REPEAT STEP 1. |
| 3. Hold up two objects - same size; weights of 2 and 5..... | REPEAT STEP 1. |
| 4. Hold up two objects - same size; same weight (4)..... | REPEAT STEP 1. |
| 5. Hold up one object..... | HOW MUCH DOES IT WEIGH?<br>HOW CAN WE FIND OUT?<br>WHAT DO WE USE TO WEIGH THINGS? |
| Continue until someone (including the teacher) states that we use a scale or balance to weigh objects. | |



End or Cont'd

Introduction of Balance

Purpose(s): a) To teach how a balance can be used to weigh objects.

Materials: Ten objects, each with a weight of 1,2,3,4,5, or 6 units.
A balance for each student - unmarked pans.
A set of 9 unit weights for each student.

Vocabulary: Use your balance to weigh it.
How much does it weigh?
What does it weigh?
It weighs five units.

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Hold up an object - weight 3..... | HOW MUCH DOES THIS WEIGH?<br>HOW CAN WE FIND OUT? |
| Place a balance on the table..... | THIS IS A BALANCE. WE CAN USE IT TO WEIGH THINGS. |
| Place object in left pan..... | FIRST, WE PUT THE OBJECT IN A PAN. |
| Hold up a unit..... | THIS IS A UNIT. |
| Add units to right pan, one at a time. Pause after adding each unit. After the 3rd unit, wait until the balance comes to rest..... | SEE, 3 UNITS MAKE IT BALANCE.<br>HOW MUCH DOES THE (OBJECT) WEIGH?<br>IT WEIGHS 3 UNITS. |
| 2. Hold up an object - weight 2..... | HOW MUCH DOES THIS WEIGH?<br>HOW CAN WE FIND OUT? |
| Let one student demonstrate the weighing process..... | WHAT DOES THE (OBJECT) WEIGH?<br>IT WEIGHS 2 UNITS. |
| 3. Hold up an object - weight 5..... | WHAT DOES THIS WEIGH?<br>(STUDENT) SHOW US HOW TO WEIGH IT. |
| Let the student demonstrate..... | HOW MUCH DOES THE (OBJECT) WEIGH?<br>IT WEIGHS 5 UNITS. |



End or Cont'd

Lesson Outline Continued

| ACTION | QUESTIONS - DIRECTIONS - TO |
|---|---|
| 4. Give each student a balance, at most 4 balances per table.<br>Place several objects on each table..... | NOW YOU CAN USE YOUR BALANCES TO WEIGH THE OBJECTS.<br><br>WHEN YOU FIND OUT HOW MUCH SOMETHING WEIGHS, RAISE YOUR HAND, AND TELL ME WHAT IT WEIGHS. |
| Let the students weigh each object; help individuals, if necessary, find the weight of an object.<br><br>If necessary, place other objects on the table to provide additional practice | |

Weight Equivalence

Purpose(s): a) To teach techniques of using a balance to determine whether or not two objects have the same weight.

Materials: Ten objects, each with a weight of 1,2,3,4,5, or 6 units.
A balance for each student - unmarked pans.
A set of 9 unit weights for each student.

Vocabulary: Do they weigh the same?
Do they have the same weight.

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Hold up two objects, same weight, different sizes..... | DO THESE WEIGH THE SAME?<br>HOW CAN WE FIND OUT? |
| Follow students' suggestions..... | SEE THEY HAVE THE SAME WEIGHT.<br>HOW DO WE KNOW THIS? |
| NOTE: It is not necessary at this point for the students to know the actual weight. | |
| 2. Give each student his balance and weights. | |
| Place 8 objects on the table, 4 pairs with objects in each pair having the same weight..... | FIND TWO OBJECTS WHICH WEIGH THE SAME. |
| Let the pupils work on this task..... | SHOW ME THE OBJECTS WHICH WEIGH THE SAME. |
| 3. Hold up two objects which have the same weight..... | DO THESE WEIGH THE SAME? |
| Show only one of the objects..... | HOW MUCH DOES THIS WEIGH?<br>HOW CAN WE FIND OUT?<br>WHAT DOES IT WEIGH? |
| Hold up the other object..... | HOW MUCH DOES THIS WEIGH?<br>HOW CAN WE FIND OUT?<br>HOW MUCH DOES IT WEIGH? |
| Hold up both objects..... | DO BOTH OF THESE WEIGH (____) UNITS?<br>HOW DO WE KNOW? |



End or Cont'd

## Introduction to Symmetrical Property

Purpose(s): a) To verify that in determining the weight of an object, the object may be placed on either side of the balance.

Materials: Balance and set of 9 units for each student - unmarked pan.
10 objects, each with a weight of 1,2,3,4,5,, or 6 units.

Vocabulary: This side of the balance.
The right side of the balance.
The left side of the balance.

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Hold up an object..... | HOW MUCH DOES THIS WEIGH? |
| Let a student find the weight..... | IT WEIGHTS (_____) UNITS. |
| Remove unit weights from pan. | |
| Move the object to the other pan..... | HOW MUCH DOES THIS WEIGH?<br>WILL IT WEIGH THE SAME? |
| Let a student weigh the object..... | DOES IT STILL WEIGH THE SAME? |
| | TO WEIGH SOMETHING, MAY WE PUT IT ON EITHER SIDE OF THE BALANCE? |
| | WILL IT WEIGH THE SAME ON EITHER SIDE? |
| 2. Give each student a balance, a set of unit weights and one object..... | PUT YOUR OBJECT ON THE LEFT SIDE.<br>HOW MUCH DOES IT WEIGH?<br>NOW, PUT IT ON THE RIGHT SIDE.<br>HOW MUCH DOES IT WEIGH?<br>DOES IT WEIGH THE SAME? |
| 3. Have the students take another object..... | PUT IT ON THE RIGHT SIDE.<br>HOW MUCH DOES IT WEIGH?<br>NOW PUT IT ON THE LEFT SIDE.<br>HOW MUCH DOES IT WEIGH?<br>DOES IT WEIGH THE SAME? |
| 4. Let the student weigh 2 or 3 other objects, using both sides of the balance. | |

End or Cont'd

Recording Weights

Purpose(s): a) To teach how to record the weight of an object.

Materials: Balance and set of 9 unit weights for each student - unmarked pans.
10 objects each with a weight of 1,2,3,4,5, or 6 units.
Record sheets - Form 1.

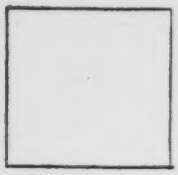
Vocabulary: Write the weight in the box.

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Place an object on the left side.<br>Have a student weigh the object.....<br>Place a record sheet in front of the balance.<br>Point to the right hand box on the record sheet..... | HOW MUCH DOES IT WEIGH?<br>(NAME) WRITE THE WEIGHT IN THIS BOX. |
| 2. Place an object on the right side.<br>Have a student weigh the object.<br>Place a record sheet in front of the balance.<br>Point to the left hand box on the record sheet..... | WRITE THE WEIGHT IN THIS BOX. |
| 3. Repeat step(s) (1) and (2) so that each student will have an opportunity to record a weigh. | |
| 4. Give each student a balance and a set of unit weights.<br>Place 5 objects on the table. Give each student 5 record sheets..... | WEIGH 5 THINGS AND WRITE THE WEIGHT IN THE BOX. |



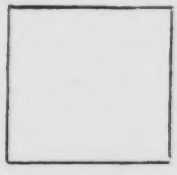
End or Cont'd



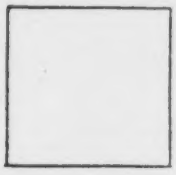
Left

Right

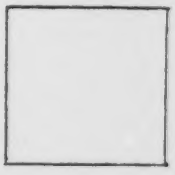
Left



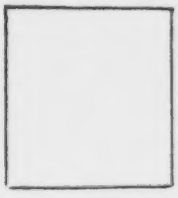
Right



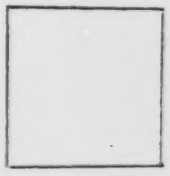
Left



Right

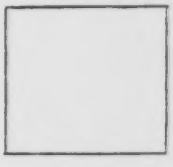


Left



Right

Left



Right

## Introduce Word Equals

Purpose(s): a) To use the word equals in oral statements to describe the relationship between two sets of weights.

Materials: None

Vocabulary: The weights are equal.

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Place 3 unit weights on left side..... | HOW MANY UNITS? HOW MUCH DO THEY WEIGH? |
| Let a student use the balance to determine (verify) the answers. | |
| Point to left side, then right side..... | DO THESE WEIGH THE SAME AS THESE? |
| | WE SAY THE WEIGHT ON THIS SIDE EQUALS THE WEIGHT ON THIS SIDE. |
| 2. Place 4 unit weights on both sides..... | ARE THE WEIGHTS EQUAL? |
| | WHY? |
| 3. Place 2 unit weights on left side and 3 unit weights on on the right side..... | ARE THE WEIGHTS EQUAL? |
| | WHY? |
| 4. Place 1 unit weight on both sides..... | ARE THE WEIGHTS EQUAL? |
| | WHY? |



End or Cont'd

## Introduce Equals Sign

Purpose(s): a) To use an equals sign in writing number sentences.

Materials: Balance for each student - unmarked pans.
12 unit weights per student.
Record sheets - Form 1 and Form 2.

Vocabulary: None.

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Give each student a record sheet - Form 1.. | WRITE THE NUMBER OF UNITS ON YOUR WORKSHEET. |
| Place 4 unit weights on right side..... | HOW MUCH DO THESE WEIGH? |
| Let a student verify answer on the balance and record weights (Form 1)..... | WRITE THE WEIGHT ON YOUR WORKSHEET.<br>ARE THE WEIGHTS EQUAL? |
| | TO SHOW THAT THE WEIGHTS ARE EQUAL WE PUT THIS SIGN BETWEEN ON THE BALANCES. |
| Put equals sign on balances..... | THIS IS AN EQUALS SIGN. |
| | COPY THE EQUALS SIGN ON YOUR WORKSHEET. |
| Demonstrate on worksheet. | |
| Point to equals sign..... | WHAT DOES THIS TELL US? |
| Point to sentence..... | THIS IS A SENTENCE. |
| | WE READ 4 EQUALS 4. |
| | NOW, READ THE SENTENCE FOR ME. |
| 2. Give each student a record sheet - Form 1.. | WRITE HOW MANY UNITS ON YOUR WORKSHEET. |
| Place 3 unit weights in left pan..... | HOW MUCH DO THESE WEIGH? |
| Have a student demonstrate on the balance and record weights..... | SHOW ME ON THE BALANCE. |

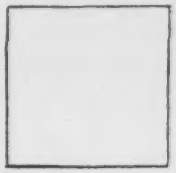
End or Cont'd

| ACTION | QUESTIONS DIRECTIONS ETC. |
|---|---|
| | WRITE THE WEIGHTS ON YOUR WORKSHEET. |
| | CAN WE PUT AN EQUALS SIGN ON THE BALANCE? |
| Put equals sign on balance..... | WHY? |
| Demonstrate..... | WRITE THE EQUALS SIGN ON YOUR PAPER. |
| | READ THE SENTENCE. |
| 3. Give each student a balance and record sheets - Form 2..... | MAKE UP SOME PROBLEMS. |
| | FOR EACH PROBLEM FILL OUT A WORKSHEET. |



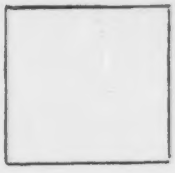
End or Cont'd

Left

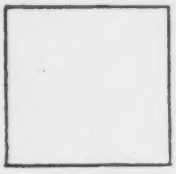


Right

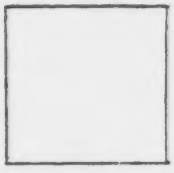
Left



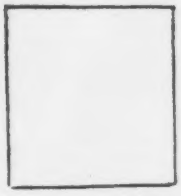
Right



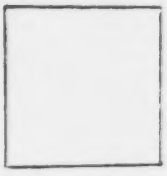
Left



Right

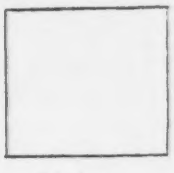


Left



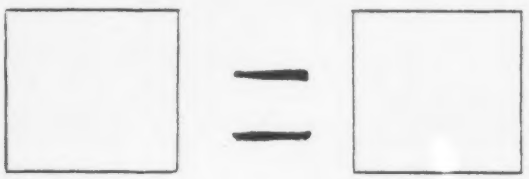
Right

70



Right

Form 2 Name _____



## Constructing Problems From Worksheets

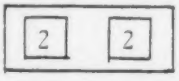
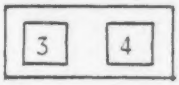
Purpose(s): a) To teach how to construct a balance situation based on information given on a worksheet.

b) To teach when to write an equals sign to show that two weights are the same.

Materials: Balance for each student - unmarked pans.
12 unit weights per child.
Worksheets E-11 - E-17.

Vocabulary: Write equals sign between the numbers.
The weights are not equal.

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Show worksheet E-11  .... | WHAT DOES THIS WORKSHEET TELL US TO DO? |
| Have a student construct the problem on the balance..... | ARE THE WEIGHTS EQUAL? |
| Put equals sign on balance..... | DO WE PUT AN EQUALS SIGN ON THE BALANCE? |
| | HOW DO WE SHOW THIS ON THE WORKSHEET? |
| Have a student write = between the boxes... | READ THIS FOR ME. |
| 2. Show worksheet E-12  .... | WHAT DOES THIS WORKSHEET TELL US TO DO? |
| Have a student construct the problem on the balance..... | ARE THE WEIGHTS EQUAL? |
| | DO WE PUT AN EQUALS SIGN ON THE BALANCE? |
| | DO WE WRITE AN EQUALS SIGN BETWEEN THE NUMBERS? |
| Use not equal in explanation..... | WHY? |
| 3. Give each student copies of worksheets E-13 - E-17..... | DO WHAT THE WORKSHEETS TELL YOU TO DO. |
| | IF THE WEIGHTS ARE EQUAL, WRITE AN EQUALS SIGN BETWEEN THE NUMBERS. |



End or Cont'd

E-11-12

Name _____

Left

Right

Left

Right

## Introduce Not Equals Sign

Purpose(s): a) To teach how to write a not equals sign to show that two weights are not the same.

Materials: Balance for each student - unmarked pans.
12 unit weights per student.
Worksheets E-18 - E-29.

Vocabulary: Write a not equals sign.

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Give worksheet E-18 [5] [5] to | |
| students..... | WHAT DOES THE WORKSHEET TELL US TO DO? |
| Have a student show on a balance..... | ARE THE WEIGHTS EQUAL? |
| Put equals sign on balance..... | DO WE PUT AN EQUALS SIGN ON THE BALANCE? |
| | WHAT DO WE WRITE BETWEEN THE NUMBERS? |
| Have students write an equals sign between the numbers. | |
| Point to sentences..... | READ THIS FOR ME. |
| 2. Give worksheet E-19 [5] [2] to | |
| students..... | WHAT DOES THE WORKSHEET TELL US TO DO? |
| Have a student show on a balance..... | ARE THE WEIGHTS EQUAL? |
| | DO WE PUT AN EQUALS SIGN ON THE BALANCE? |
| | DO WE WRITE AN EQUALS SIGN BETWEEN THE NUMBERS? |
| Put not equals sign on balance. | |
| Point to sign..... | THIS IS A NOT EQUAL SIGN. |
| Point to sentence..... | WRITE THE NOT EQUALS SIGN ON YOUR PAPER. |

End or Cont'd

| ACTION | QUESTIONS DIRECTIONS ETC. |
|---|---|
| | WE READ 5 IS NOT EQUAL TO 3. NOW, YOU READ THE SENTENCE. |
| 3. Give students worksheets E-20 - E-29..... | DO WHAT THE WORKSHEETS TELL YOU TO DO.<br><br>IF THE WEIGHTS ARE EQUAL, WRITE AN EQUALS SIGN BETWEEN THE NUMBERS.<br><br>WRITE A NOT EQUALS SIGN IF THE WEIGHTS ARE NOT THE SAME. |



End or Cont'd

Name _____

Left

Right

Left

Right

## Practice Writing Number Sentences

Purpose(s): a) To teach how to write a number sentence which corresponds to a given situation on a balance.

Materials: Balance for each student - unmarked pans.
12 unit weights per student.
Record Sheets - Form 1.

Vocabulary: None.

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Give each child a record sheet - Form 1.... | WHAT DO WE WRITE IN THE BOXES. |
| Place 3 unit weights in both pans..... | ARE THE WEIGHTS EQUAL? |
| | HOW DO WE SHOW THIS ON THE BALANCE? |
| | WHAT DO WE WRITE BETWEEN THE NUMBERS? |
| Have each child complete a record sheet. | |
| | READ THIS SENTENCE FOR ME. |
| 2. Place 4 unit weights in left pan; 1 unit weight in right pan..... | WHAT DO WE WRITE IN THE BOXES? |
| | ARE THE WEIGHTS EQUAL? |
| | HOW DO WE SHOW THIS ON THE BALANCE? |
| Have each child complete a record sheet.... | WHAT DO WE WRITE BETWEEN THE NUMBERS? |
| | READ THE SENTENCE FOR ME. |
| 3. Continue, using the following pairs: (2, 5); (4, 4); (2, 3); (3, 2); (5, 6); (6, 6). | |



End or Cont'd

## Truth-Value of Statements

Purpose(s): a) To teach techniques for determining whether a statement is true or false.

Materials: For each student: a balance, unmarked pans, 12 units.
Worksheets E-30 - E-36.

Vocabulary: Is the sentence correct?
Is the sentence wrong?

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Show worksheet E-30..... | READ THIS FOR ME. |
| | IS THE SENTENCE CORRECT? |
| Demonstrate on balance..... | HOW DO WE KNOW? |
| | HOW CAN WE SHOW THAT IT IS CORRECT? |
| Agree on some method for showing that the statement is true: circle the equation; write a C or ✓ beside the equations. Have students mark their worksheets. | |
| 2. Show worksheet E-31..... | READ THIS FOR ME. |
| | IS THE SENTENCE CORRECT? |
| Demonstrate on balance..... | HOW DO WE KNOW? |
| | HOW CAN WE SHOW THAT IT IS WRONG? |
| Possible method: write an X beside the equation. Have students mark their worksheets. | |
| 3. Give each student worksheets E-32-36..... | HERE ARE SOME SENTENCES. |
| | WHICH ONES ARE CORRECT? WHICH ARE WRONG? |



End or Cont'd

| ACTION | QUESTIONS DIRECTIONS ETC. |
| --- | --- |
| | YOUR BALANCE CAN HELP YOU FIND OUT.<br><br>HOW WILL YOU SHOW THAT A SENTENCE IS CORRECT: THAT IT IS WRONG? |





End or Contd

Name _____

$5 = 5$

$4 = 6$

Introduction to Addition

Purpose(s): a) To introduce 2-region pan.
b) To introduce the union of two sets.

Materials: For each student: balance; unmarked pan left, 2-region pan right; 6 unit weights.
Objects to weigh.

Vocabulary: How many units in the blue part?
How many units in the yellow part?
How many units all together?

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Point to 2-region pan. ..... | THIS IS NEW.<br>HOW IS IT DIFFERENT? |
| Have students identify blue and yellow parts. Observe that there are 2 parts. | |
| 2. Place an object on unmarked pan. ..... | HOW MUCH DOES IT WEIGH? |
| Place some units in blue region and some in yellow region to make the scale balance. | HOW MANY UNITS ARE IN THE BLUE PART?<br>HOW MANY IN THE YELLOW PART?<br>HOW MANY UNITS ALTOGETHER?<br>HOW MUCH DOES IT WEIGH? |
| 3. Have each student place an object on unmarked pan. ..... | HOW MUCH DOES IT WEIGH?<br>PUT SOME UNITS IN THE BLUE PART, AND SOME IN THE YELLOW PART. |
| Have each student describe the balance. ... | HOW MANY UNITS IN THE BLUE PART?<br>HOW MANY UNITS IN THE YELLOW PART?<br>HOW MANY UNITS ALTOGETHER?<br>HOW MUCH DOES IT WEIGH? |
| 4. Repeat 3, using another object. | |

Recording Units in 2-Region Pan

Purpose(s): (a) To teach recording of the number of units in each region.
(b) To introduce zero.

Materials: For each student: balance; unmarked pan left, 2-region pan right.
Form 3
Objects to weigh.

Vocabulary: There are zero units in this part.

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Give each student a copy of Form 3.<br>Place an object on unmarked pan.<br>Place several units in first region.<br>(Left part on 2-region pan),<br>But not enough to balance. .....<br>Demonstrate recording. .....<br><br>Place units in second region to obtain a balance. .....<br>Demonstrate. ..... | <br><br><br><br>HOW MANY UNITS IN THIS PART?<br>WRITE THE NUMBER ON YOUR WORKSHEET IN THIS BOX.<br>DOES IT BALANCE?<br><br>HOW MANY UNITS IN THIS PART?<br>WRITE THE NUMBER IN THIS BOX.<br>HOW MANY UNITS ALTOGETHER?<br>HOW MUCH DOES IT WEIGH? |
| 2. Repeat (1) using another object. | |
| 3. Repeat (1) using another object, except in the first step use enough units to obtain a balance.<br>State that there are zero units in second region. | |
| 4. Give each student 5 copies of Form 3.<br>Have them weigh an object and record the number of units in each region. ..... | <br><br>WEIGH FIVE THINGS. EACH TIME<br>WRITE THE NUMBER OF UNITS IN EACH PART ON YOUR WORKSHEET. |



End or Cont'd

The Addition Sign

Purpose(s): (a) To introduce the addition sign.

Materials: For each student: balance, unmarked pan left, 2-region pan right.
6 units.
Forms 3 and 4.
Objects to weigh.

Vocabulary: We read this: Three Plus Two.

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Give each student a copy of Form 3. Place object in un-marked pan. ..... | WEIGH THIS FOR ME. REMEMBER, THE UNITS GO IN EITHER PART, NOT ON THE LINE. |
| Let a student weigh the object. | FILL OUT YOUR WORKSHEET.<br>HOW MANY UNITS IN BLUE PART?<br>HOW MANY UNITS IN YELLOW PART?<br>HOW MANY UNITS ALTOGETHER?<br>HOW MUCH DOES IT WEIGH? |
| Write + between the boxes; and have each student do the same. ..... | TO SHOW THAT WE WANT TO KNOW HOW MANY ALTOGETHER, WRITE THIS SIGN BETWEEN THE BOXES. |
| Pointing to worksheet, ..... | THIS IS A PLUS SIGN. WE READ THIS (__) PLUS (__).<br>NOW, YOU READ IT FOR ME. |
| 2. Repeat (1) using another object. | |
| 3. Give each student a copy of Form 4. Have each student weigh an object and record weights. ..... | WEIGH SOMETHING ON YOUR BALANCE.<br>WRITE THE NUMBER OF UNITS ON YOUR WORKSHEET. |



End or Cont'd

| ACTION | QUESTIONS DIRECTIONS ETC. |
|---|---|
| In turn, pointing to worksheet, ask each child..... | READ THIS FOR ME.<br>HOW MANY UNITS IN BLUE PART?<br>HOW MANY UNITS IN YELLOW PART?<br>HOW MANY UNITS ALTOGETHER?<br>HOW MUCH DOES IT WEIGH? |
| 4. Give each student five copies of Form 4. .... | WEIGH FIVE THINGS.<br>WRITE THE NUMBER OF UNITS ON YOUR WORKSHEET.<br>TELL ME HOW MUCH EACH THING WEIGHS. |

Addition Equation

Purpose(s): (a) To introduce addition equations.

Materials: For each student: balance, unmarked pan left, 2-region pan right, 12 unit weights.
Form 5.

Vocabulary: 5 equals 2 plus 3.

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Give each student a copy of Form 5. Put 5 unit weights on unmarked pan. ..... | HOW MANY UNITS? |
| Point to box on left side of equal sign. .. | WRITE THE NUMBER ON YOUR WORKSHEET. |
| Put 2 units on blue part. ..... | HOW MANY UNITS? |
| Point to first box, right side of equation. | WRITE THE NUMBER ON YOUR WORKSHEET. IS IT EQUAL? |
| Put 3 units on yellow part. ..... | HOW MANY UNITS? |
| Point to second box, right side of equation. | WRITE THE NUMBER ON YOUR WORKSHEET. NOW IS IT EQUAL? |
| Point to 2-region pan. ..... | WE CAN PUT THE EQUAL SIGN ON THE BALANCE. HOW MANY UNITS ON THIS SIDE? DOES 5 EQUAL 2 PLUS 3? |
| Point to equation 5=2+3. ..... | WE READ THIS... FIVE EQUALS TWO PLUS THREE. NOW YOU READ IT. |
| 2. Repeat (1) to show 3=2+1. | |
| 3. Give each student a copy of Form 5. ..... | PUT 6 UNITS ON LEFT SIDE. WRITE THE NUMBER ON YOUR WORKSHEET. PUT 4 UNITS ON BLUE PART. WRITE THE NUMBER ON YOUR WORKSHEET. |

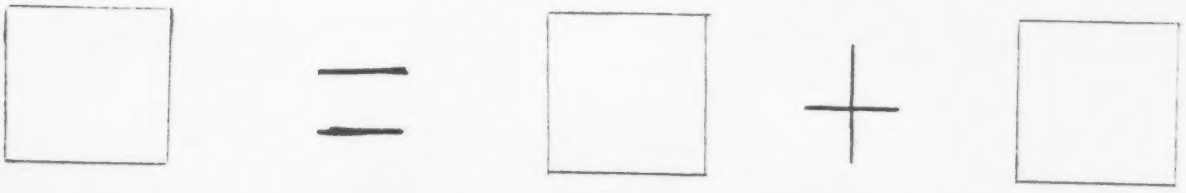


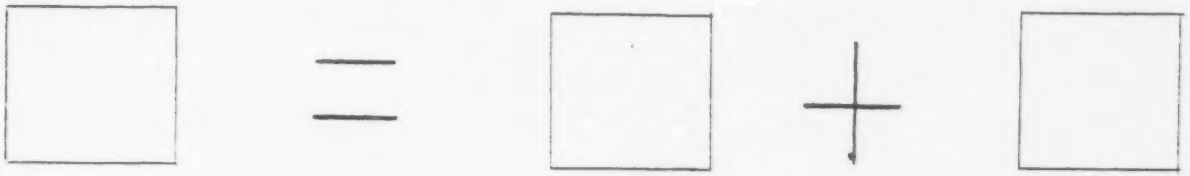
End or Cont'd

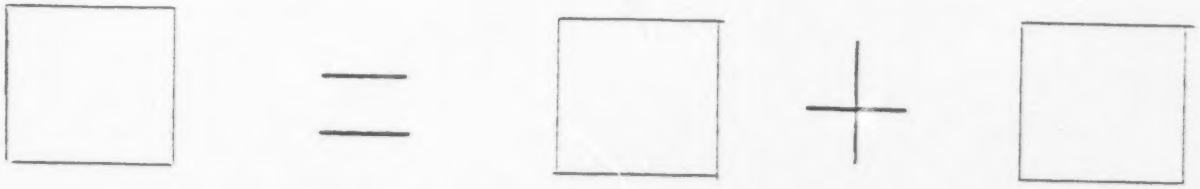
| ACTION | QUESTIONS DIRECTIONS ETC. |
|---|---|
| | IS IT EQUAL?<br>PUT 2 UNITS ON YELLOW PART.<br>WRITE THE NUMBER ON YOUR WORKSHEET.<br>IS IT EQUAL?<br>PUT THE EQUAL SIGN ON YOUR BALANCE.<br>DOES 6 EQUAL 4 PLUS 2?<br>READ THE SENTENCE FOR ME. |
| 4. Repeat (3) to show 4=2+2. | |
| 5. Give each student a copy of Form 5. ..... | PUT 3 UNITS ON LEFT SIDE.<br>WRITE THE NUMBER.<br>PUT 3 UNITS ON BLUE PART.<br>WRITE THE NUMBER.<br>IS IT EQUAL?<br>PUT THE EQUAL SIGN ON YOUR BALANCE.<br>HOW MANY UNITS DO WE PUT ON THE YELLOW PART?<br>WHAT DO WE WRITE?<br>DOES 3=3+0?<br>READ THE SENTENCE FOR ME. |
| 6. Let students make up 5 problems and record answers on Form 5. | |

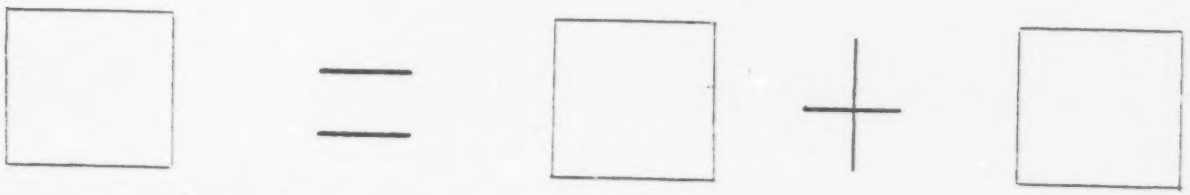
Form 5

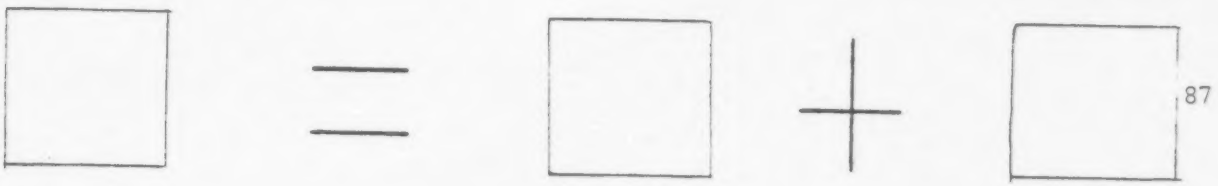
Name _____











Finding Combinations of Weights

Purpose(s): (a) To teach that a number may be expressed as the sum of more than one pair of numbers.

Materials: For each student: balance, unmarked pan left, 2-region pan right, 12 unit weights.
Form 5.

Vocabulary:

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Give each child 3 copies of Form 5. | |
| Place 5 unit weights on un-marked pan. .... | HOW MANY UNITS?<br>WRITE THE NUMBER ON YOUR WORKSHEET. |
| Place 4 units on blue part. ..... | HOW MANY UNITS ON BLUE PART?<br>WRITE THE NUMBER.<br>IS IT EQUAL? |
| Place 1 unit on yellow part. ..... | HOW MANY UNITS ON YELLOW PART?<br>WRITE THE NUMBER.<br>IS IT EQUAL?<br>WE NOW PUT THE EQUAL SIGN ON THE BALANCE.<br>DOES 5 EQUAL 4 PLUS 1?<br>READ THE SENTENCE FOR ME. |
| Remove units from right side. ..... | LET'S FIND ANOTHER WAY TO MAKE IT EQUAL.<br>HOW MANY UNITS ON LEFT SIDE?<br>WRITE THE NUMBER. |
| Place 3 units on blue part. ..... | HOW MANY UNITS ON BLUE PART?<br>WRITE THE NUMBER.<br>IS IT EQUAL? |
| Place 2 units on yellow part. ..... | HOW MANY UNITS ON YELLOW PART?<br>WRITE THE NUMBER.<br>IS IT EQUAL?<br>WE CAN PUT THE EQUAL SIGN ON THE BALANCE.<br>DOES 5 EQUAL 3 PLUS 2?<br>READ THE SENTENCE. |
| Remove units from right side.<br>Remove equals sign. ..... | PUT 5 UNITS ON THE LEFT SIDE OF YOUR BALANCE. |



End or Cont'd

| ACTION | QUESTIONS DIRECTIONS ETC. |
|---|---|
| Show completed forms. ..... | WRITE 5 ON YOUR WORKSHEET.<br><br>WE HAVE TWO WAYS TO MAKE THE SIDES EQUAL.<br>WHAT ARE THEY?<br>NOW FIND ANOTHER WAY TO MAKE IT EQUAL.<br>WRITE THE NUMBERS ON YOUR WORKSHEET. |
| Have students leave their solutions on balance. Ask students to show their equation and read it. | |
| 2. Repeat (1) using 4 units on left pan. Include in demonstration that 4=0+4. | |
| 3. Repeat (1) using 1 unit in left pan. | |
| 4. Repeat (1) using 2 units in left pan. | |
| 5. Repeat (1) using 3 units in left pan. | |
| 6. Repeat (1) using 6 units in left pan. | |

Constructing Balance Problems.

Purpose(s): a) To teach techniques for constructing a model on a balance to correspond to a given equation.

Materials: For each student: balance, unmarked pan left, 2-region pan right, 12 unit weights.
Worksheets A1 - A7

Vocabulary:

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Give worksheet A1 to each student. . . . | READ THIS SENTENCE FOR ME. |
| Place A1 in front of balance<br>Point to unmarked pan. . . . . | HOW MANY UNITS DO WE PUT ON THIS SIDE? |
| Put 5 units on unmarked pan.<br>Point to blue region. . . . . | HOW MANY UNITS DO WE PUT ON THE BLUE PART? |
| Put 1 unit in blue region.<br>Point to yellow region. . . . . | HOW MANY UNITS DO WE PUT ON THE YELLOW PART? |
| Put 4 units in the yellow region. . . . | DOES IT BALANCE?<br>PUT AN EQUAL SIGN ON THE BALANCE.<br>DID WE MAKE THE BALANCE LOOK LIKE THE SENTENCE? |
| 2. Give worksheet A2 to each student. . . . | READ THIS SENTENCE FOR ME.<br>NOW MAKE YOUR BALANCE LOOK LIKE THE SENTENCE. |
| Check each pupil's work. | |
| 3. Give worksheets A3-7 to students. . . . | HERE ARE SOME SENTENCES.<br>WHAT DO THE PROBLEMS TELL US TO DO?<br>USE YOUR BALANCE TO SHOW EACH SENTENCE. |



End or Cont'd

A-1-2 Name _____

$5 = 1 + 4$

$6 = 4 + 2$

Completing Equations: Missing Addends

Purpose(s): a) To provide practice an finding two numbers whose sum is a given number.

Materials: For each student: balance; unmarked pan left, 2-region pan right, 12 unit weights.

Worksheets A8-A14

Vocabulary:

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Give a copy of A8 to each student. . . . | WHAT DOES THIS WORKSHEET TELL US TO DO FIRST? |
| Put 6 units on unmarked pan. . . . . | NOW, HOW MANY UNITS DO WE PUT ON THE BLUE REGION? |
| Follow student suggestion. | |
| Demonstrate. . . . . | WRITE THIS NUMBER ON YOUR WORKSHEET. |
| | HOW MANY UNITS ON THE YELLOW PART TO MAKE IT BALANCE. |
| Follow student suggestion. | |
| | DOES IT BALANCE? |
| Demonstrate. . . . . | WRITE THE NUMBERS.<br>CAN WE PUT THE EQUAL SIGN ON THE BALANCE? |
| Point to equation. . . . . | NOW, READ THIS FOR ME. |
| 2. Give a copy of A9 to each student. . . . | WHAT DOES THIS WORKSHEET TELL US TO DO FIRST? |
| | SHOW ME ON YOUR BALANCE. |
| | NOW PUT SOME UNITS ON THE BLUE PART.<br>WRITE THE NUMBERS. |

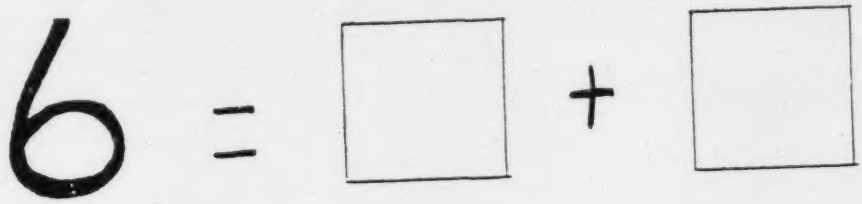


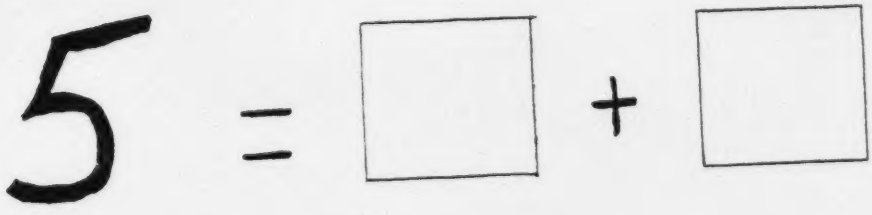
End or Cont'd

| ACTION | QUESTIONS DIRECTIONS ETC |
|---|---|
| Ask each student to read his sentence. | DOES IT BALANCE?<br>IF NOT, PUT SOME UNITS ON THE YELLOW PART.<br>DOES IT BALANCE?<br>WRITE THE NUMBER.<br>PUT THE EQUAL SIGN ON THE BALANCE. |
| 3. Give copies of A10-14 to students. . . . . | HERE ARE SOME WORKSHEETS.<br>DO WHAT THE WORKSHEETS TELL YOU TO DO. |

A-8-9

Name _____





Completing Equations: Finding Sums

Purpose(s): a) To provide practice on finding the sum of two numbers.

Materials: For each student: balance, unmarked pans on left; 2-region pan on right; 12 units.
Worksheets A15-A21.

Vocabulary: None.

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Give a copy of A-15 to each student..... | WHAT DOES THE WORKSHEET TELL US TO DO FIRST? |
| Construct model on balance..... | NOW, WHAT DO WE DO? |
| Follow student suggestions. | |
| Point to box on left side of equation..... | DOES IT BALANCE?<br>CAN WE PUT THE EQUAL SIGN ON THE BALANCE?<br>WHAT DO WE WRITE IN THIS BOX? |
| Demonstrate. | |
| Point to sentence..... | READ THIS FOR ME. |
| 2. Give a copy of A-16 to each student..... | WHAT DOES THE WORKSHEET TELL US TO DO FIRST?<br><br>SHOW ME ON YOUR BALANCE.<br>DOES IT BALANCE?<br>PUT THE EQUAL SIGN ON THE BALANCE.<br>READ THE SENTENCE FOR ME. |
| 3. Give copies of A-17-A-21 to students..... | HERE ARE SOME WORKSHEETS.<br>COMPLETE THE SENTENCES. |

Name _____

Using = and ≠ In Addition Sentences

Purpose(s): a) To provide practice on using either the relation equals or not equals to complete an addition sentence.

Materials: For each student: balance, unmarked pan on left; 2-region pan on right; 12 units.
Worksheets A-22-A-30.

Vocabulary: None.

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Give a copy of A-22 to each student..... | WHAT DOES THIS WORKSHEET TELL US TO DO? |
| Demonstrate on balance..... | DOES IT BALANCE?<br>WHAT SIGN DO WE PUT ON THE BALANCE?<br>WHAT DO WE WRITE IN THE CIRCLE BETWEEN THE NUMBERS? |
| Demonstrate..... | READ THIS SENTENCE FOR ME. |
| 2. Repeat (1), using A-23. | |
| 3. Repeat (1), using A-24. | |
| 4. Repeat (1), using A-25. | |
| 5. Give worksheets A-26-A-30 to students..... | DO WHAT THE WORKSHEETS TELL YOU TO DO.<br>COMPLETE THE SENTENCES. |

A-22-25

Name _____

3 ○ 3

5 ○ 2

6 ○ 4 + 2

4 ○ 2 + 3

Introduction to Missing Addends

Purpose(s): (a) To teach the meaning of misssing addend sentences and techniques for solving such problems.

Materials: For each student: balance, unmarked pan left, 2-region pan right; 12 units.
Worksheets: MA1-2

Vocabulary: How many did we add to make it equal?

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Set up balance so that there are 6 units on left and 2 units on the blue region...<br><br>Follow student suggestion, but show adding 4 units to yellow region. | ARE THE WEIGHTS EQUAL?<br>HOW DO WE KNOW?<br>HOW CAN WE MAKE THEM EQUAL? |
| 2. Show worksheet MA-1a.....<br><br>Demonstrate. | WHAT DOES THIS WORKSHEET TELL US TO DO?<br><br>SET UP YOUR BALANCES.<br><br>NOW MAKE IT EQUAL.<br>HOW MANY DID YOU ADD TO MAKE IT EQUAL?<br>WHAT DO WE WRITE IN THIS BOX?<br>READ THE SENTENCE. |
| 3. Repeat (2) using worksheet MA-1b. | |
| 4. Give worksheets MA 2 to students..... | COMPLETE THE WORKSHEETS USING YOUR BALANCE TO MAKE SURE YOU ARE RIGHT. |



End or Cont'd

MA 2

Name_____

| | |
|---|---|
| $6 = 4 + \square$ | $\square = 3 + 1$ |
| $5 = \square + 2$ | $6 = \square + \square$ |
| $5 = 2 + \square$ | $3 = 2 + \square$ |
| $3 = \square + 3$ | $3 = \square + 1$ |
| $4 = 0 + \square$ | $4 = 2 + \square$ |

100

105

Introduce 3-Region Pan

Purpose(s): (a) To introduce the 3-region pan in context of solving missing addend problems.

Materials: For each student: Balance, unmarked pan on left; 3-region pan right; 12 units.
Worksheet MA6-7

Vocabulary:

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Give each student a copy of MA6..... | READ THE SENTENCE FOR ME.<br>WHAT DOES THIS PROBLEM TELL US TO DO?<br>WHAT DO WE NEED? |
| Give each student a 3-region pan..... | NOW SHOW ME ON THE BALANCE THE PROBLEM.<br>HOW MANY DO WE ADD TO THE RED TO MAKE IT EQUAL?<br>WHAT DO WE WRITE IN THE BOX?<br>READ THE SENTENCE.<br>DOES 5 EQUAL 2 PLUS 1 PLUS 2?<br>WHY? |
| 2. Repeat (1) using second problem on MA6. | |
| 3. Repeat (1) using third problem on MA6. | |
| 4. Give each student a copy of MA7..... | COMPLETE THE PROBLEM USING YOUR BALANCE TO MAKE THEM RIGHT. |

End or Cont'd

MA6

Name _____

$5 = 2 + 1 + \square$

$4 = 1 + \square + 1$

$6 = \square + 2 + 1$

## Introduction to Subtraction

Purpose(s): (a) To introduce the concept of take away.

Materials: For each student: balance, unmarked pans; subtraction tray; 12 units.

Vocabulary: Take away.

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Put 5 units on R side.....<br>Move two units to tray.....<br><br><br>Demonstrate. | HOW MANY UNITS<br>IF I TAKE AWAY 2?<br>HOW MANY DO I PUT ON LEFT SIDE TO MAKE IT EQUAL?<br><br>SEE, 3 EQUALS 5 TAKE AWAY 2. |
| 2. Repeat (1) to show 1 = 4 - 3. | |
| 3. Have each student put 4 units on right side.....<br><br>Let students perform task. | IF YOU TAKE AWAY 1, HOW MANY WILL YOU PUT ON THE LEFT SIDE TO MAKE IT EQUAL?<br><br>SEE 1 EQUALS 4 TAKE AWAY 3. |
| 4. Repeat (3) to show 5-3, 6-3, 4-2, 3-3. | |



End or Cont'd

Subtraction Equations

Purpose(s): (a) To teach how to write subtract equations.

Materials: For each student: Balance; unmarked pans, subtraction tray; 12 units. Form 8.

Vocabulary:

LESSON OUTLINE

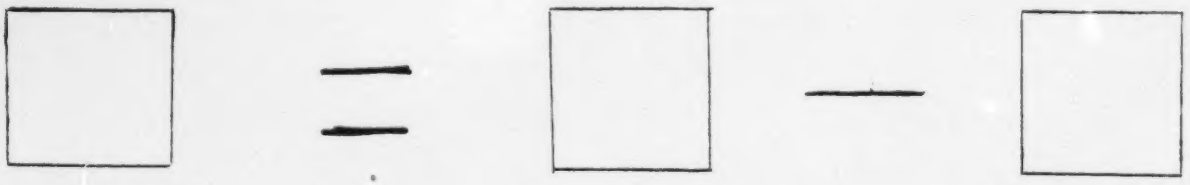
| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Give each student of copy of Form 8. | |
| Put 4 units on right side..... | HOW MANY? |
| Demonstrate..... | WHERE DO WE WRITE THE NUMBER?<br>NOW TAKE AWAY 1. |
| Demonstrate..... | WHERE DO WE SHOW THIS?<br>HOW MANY DO WE PUT ON THE LEFT SIDE TO MAKE IT EQUAL. |
| Demonstrate..... | WHERE DO WE WRITE THIS NUMBER?<br>READ THE SENTENCE. |
| 2. Repeat (1) to show 5 - 3, 6 - 1, 2 - 1, 4 - 4. | |

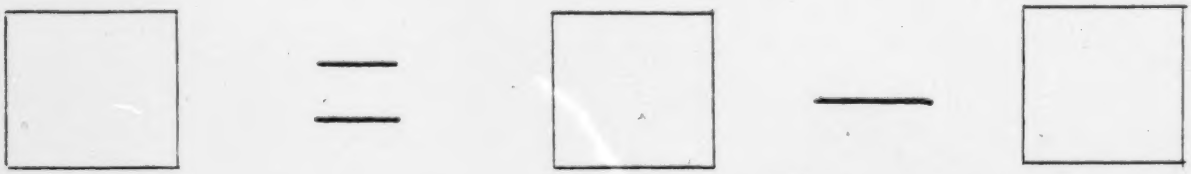


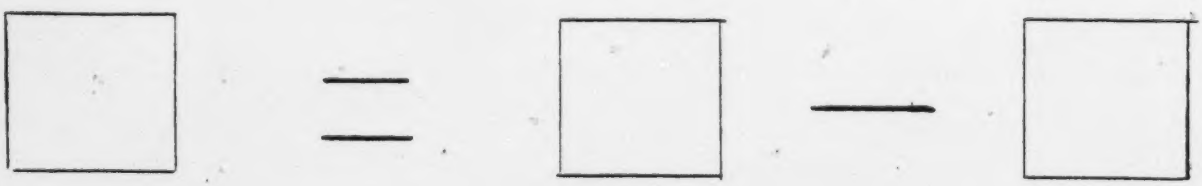
End or Cont'd

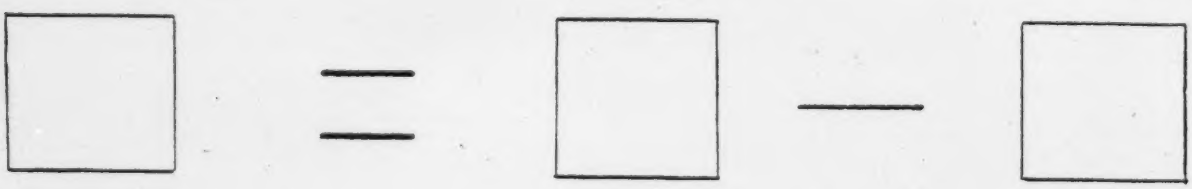
Form 8

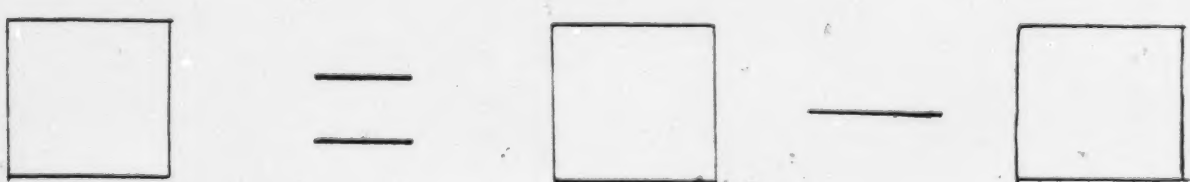
Name _____











## Constructing Balance Models

Purpose(s): (a) To drill on the meaning of subtraction sentences, by demonstrating take away situations on the balance.

Materials: For each student: Balance, unmarked pans, subtraction tray, 12 units. Worksheet S1.

Vocabulary: Show me this take away sentence on the balance.

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Show first equation on S1..... | READ THIS.<br>SHOW ME THIS TAKE AWAY SENTENCE ON YOUR BALANCE. |
| 2. Repeat (1) using other equations on S1. | |



End or Cont'd

$5 = 6 - 1$

$2 = 5 - 3$

$4 = 4 - 0$

$0 = 3 - 3$

## Solving Subtraction Problems

Purpose(s): (a) To teach a technique for finding the difference.
(b) To provide drill on solving subtraction problems.

Materials: For each student: Balance; unmarked pans; subtraction tray; 12 units.
Worksheets S2-4.

Vocabulary:

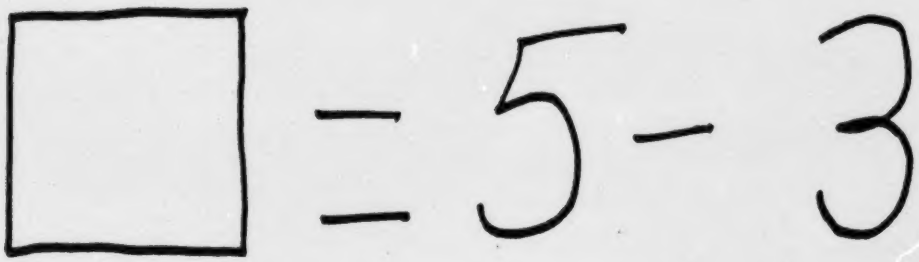
LESSON OUTLINE

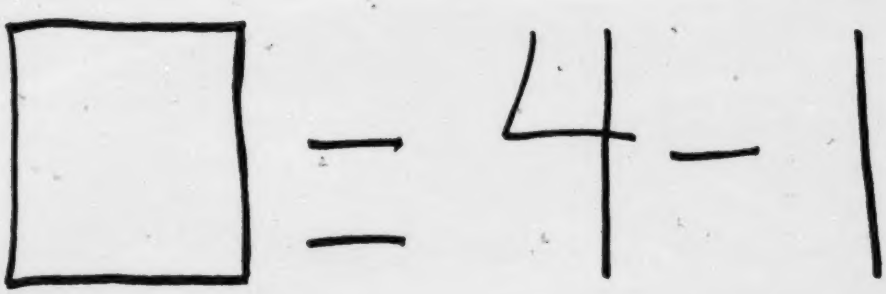
| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Give each student a copy of S2.<br>Show first equation on S2.....<br>Have each student demonstrate on scale.....<br>Write difference in box. | <br>WHAT DOES THIS PROBLEM TELL US TO DO?<br>WHAT DO WE WRITE IN THIS BOX?<br><br>READ THIS SENTENCE. |
| 2. Repeat (1) using second equation on S2. | |
| 3. Have students complete S3. | |
| 4. Have students complete S4. | |



End or Cont'd

S2 Name _____





## Equal Sign in Subtraction Equations

Purpose(s): (a) To provide drill on using equals or not equals to complete subtraction equations.

Materials: For each student: Balance; unmarked pans; subtraction tray; 12 units.
Worksheets S5 and 6.

Vocabulary:

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Show first problem on S5. ..... | WHAT DOES THIS PROBLEM TELL US TO DO? |
| Demonstrate on balance. | |
| Hold up = and ≠ signs. ..... | DO WE PUT THE = OR ≠ ON THE BALANCE? |
| | WHAT DO WE WRITE IN THE CIRCLE? |
| | READ THE SENTENCE. |
| 2. Repeat (1) using the second problem on S5. | |
| 3. Have students complete S6. | |



End XXXXXXXXX

S5 Name _____

$5 \bigcirc 6 - 1$

$2 \bigcirc 5 - 3$

## Checking Subtraction

Purpose(s): (a) To provide an opportunity for the students to reflect on the meaning of subtraction sentences.

Materials: For each student: Balance; unmarked pans; subtraction tray; 12 units.
Worksheets ~~314 and 15.~~ RT 7 & 8

Vocabulary: Is this subtraction sentence correct?

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Show the first problem on ~~314~~ RT 7. ..... | READ THIS SENTENCE.<br>IS THIS SENTENCE CORRECT?<br>HOW DO WE KNOW?<br>SHOW ME ON YOUR BALANCE. |
| Agree on some marking system - - check or star. ..... | HOW DO WE SHOW ON THE PAPER THAT IT IS CORRECT? |
| 2. Show the second problem on ~~314~~ RT 7. ..... | READ THIS SENTENCE.<br>IS IT CORRECT?<br>HOW DO YOU KNOW?<br>SHOW ME ON YOUR BALANCE. |
| Agree on some marking system - - cross. ..... | HOW DO WE SHOW ON THE PAPER THAT IT IS NOT CORRECT? |
| 3. Have students complete ~~315~~ RT 8. ..... | CHECK EACH SENTENCE.<br>ON YOUR PAPER, SHOW THAT IT IS CORRECT OR NOT CORRECT. |



End XXXXXXXXX

S14

Name _____

$4 = 6 - 2$

$1 = 5 - 3$

## Relating Subtraction Equations to Models

Purpose(s): (a) To provide an opportunity for the students to reflect on whether or not an equation and a balance model are appropriately related.

Materials: For each student: Balance; unmarked pans; 12 units.
Worksheets RT9-10.

Vocabulary:

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Give each student a copy of RT9.<br>Set up balance to show 3 = 5 - 2..... | READ THE SENTENCE.<br>DOES THE SENTENCE SHOW WHAT I DID ON THE BALANCE?<br>WHY? |
| 2. Set up balance to show 1 = 4 - 3..... | READ THE SENTENCE.<br>DOES THE SENTENCE SHOW WHAT I DID ON THE BALANCE?<br>WHY?<br>WHAT DO WE CHANGE TO MAKE IT CORRECT? |
| 3. Set up balance to show 2 = 5 - 3..... | READ THE SENTENCE.<br>DOES THE SENTENCE SHOW WHAT I DID ON THE BALANCE?<br>WHY?<br>WHAT DO WE CHANGE TO MAKE IT CORRECT? |
| 4. Set up balance to show 1 = 6 - 4..... | READ THE SENTENCE.<br>DOES THE SENTENCE SHOW WHAT I DID ON THE BALANCE?<br>WHY?<br>WHAT DO WE CHANGE TO MAKE IT CORRECT? |



End or Cont'd

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 5. Give each student a problem from RT10.....<br><br>Set Up 5 - 3, 5 - 1, 6 - 2, 6 - 4, 4 - 3, 4 - 1, 6 - 3. | WHO HAS THE SENTENCE WHICH SHOWS THIS PROBLEM? |



End or Cont'd

Name _____

$3 = 5 - 2$

$2 = 4 - 2$

$1 = 5 - 3$

$2 = 6 - 4$

Sums Families: Seven and Eight

Purpose(s): (a) To introduce sums of seven and eight.
(b) To provide drill on finding number pairs which have a sum of seven or eight.
(c) To provide additional drill with sums six or less.

Materials: For each student: Balance; unmarked pan left; 2 or 3-region pan right; 16 units.
Worksheets: A121-126.

Vocabulary:

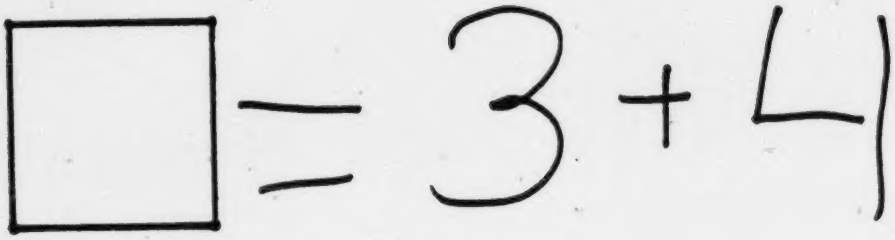
LESSON OUTLINE

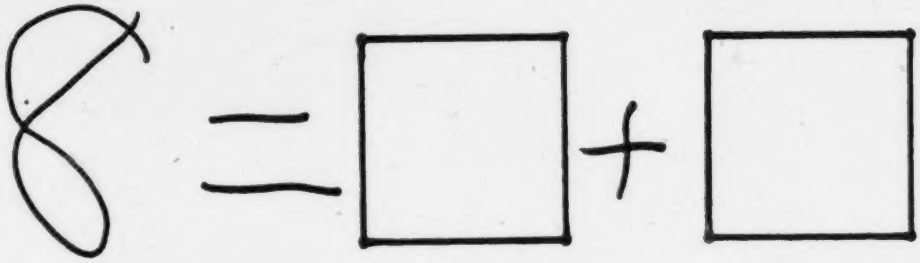
| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Show the first problem on A121.....<br><br>Demonstrate on balance. | READ THE SENTENCE.<br>WHAT DOES THE WORKSHEET TELL US TO DO?<br><br>HOW DO WE COMPLETE THE SENTENCE?<br>READ THE SENTENCE. |
| 2. Repeat (1) using the other problems on A121. | |
| 3. Have students complete worksheets A122-126. | |

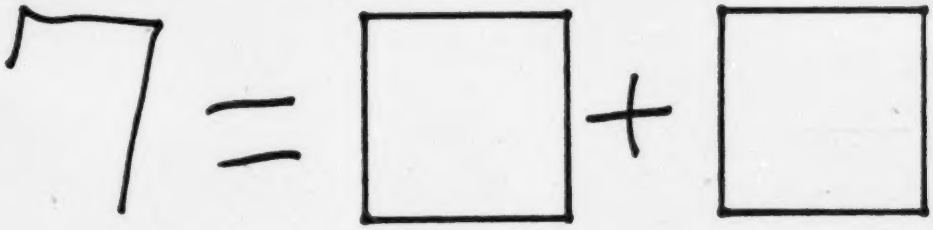


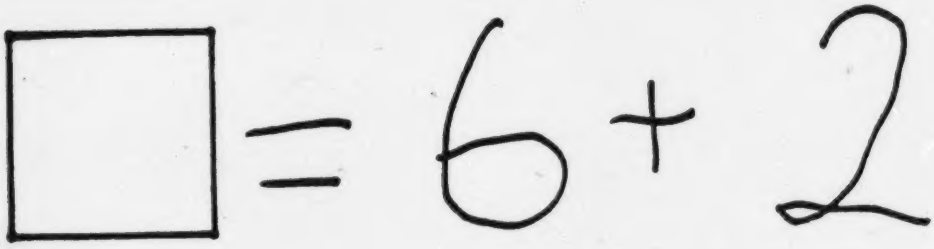
End or Cont'd

Name _____

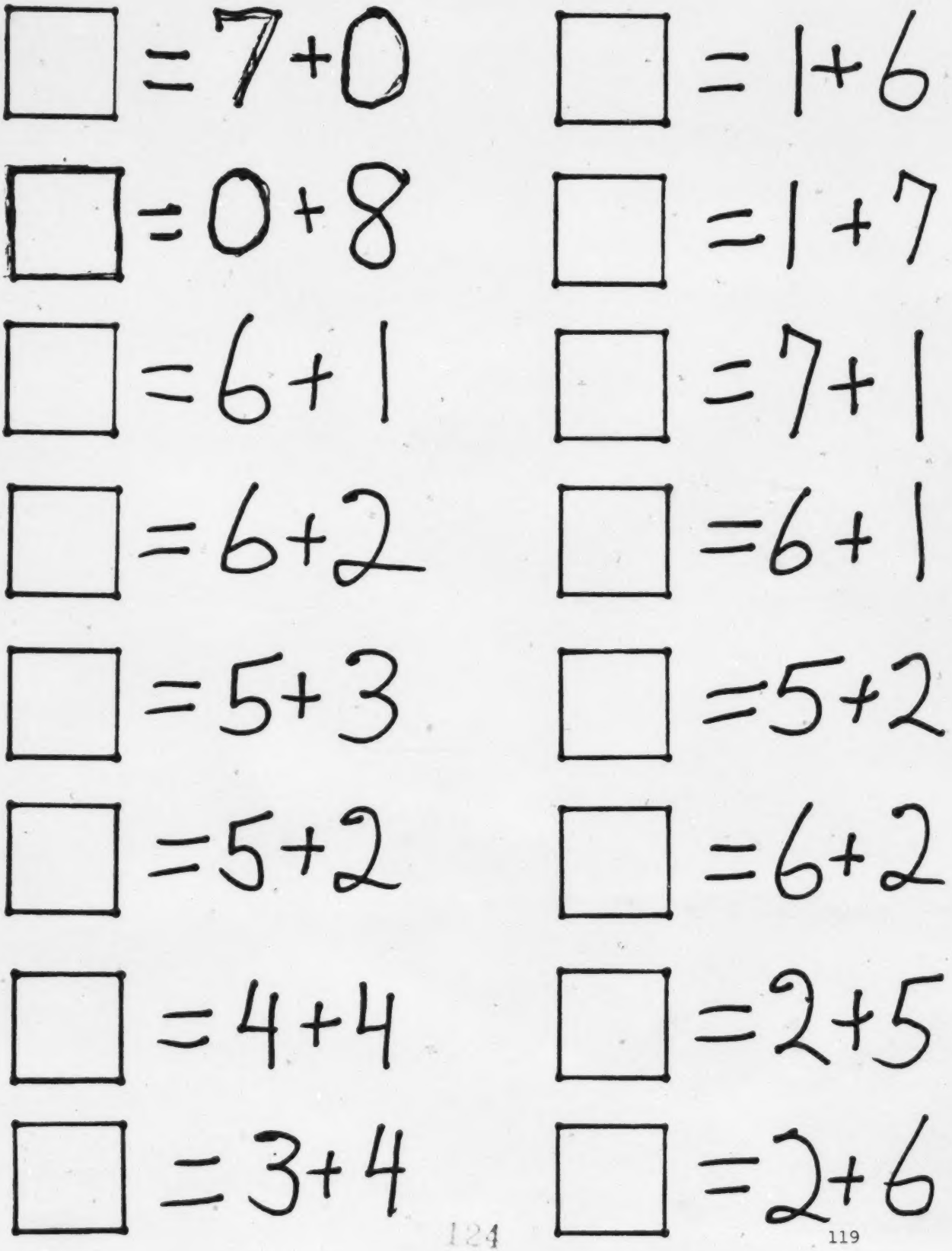




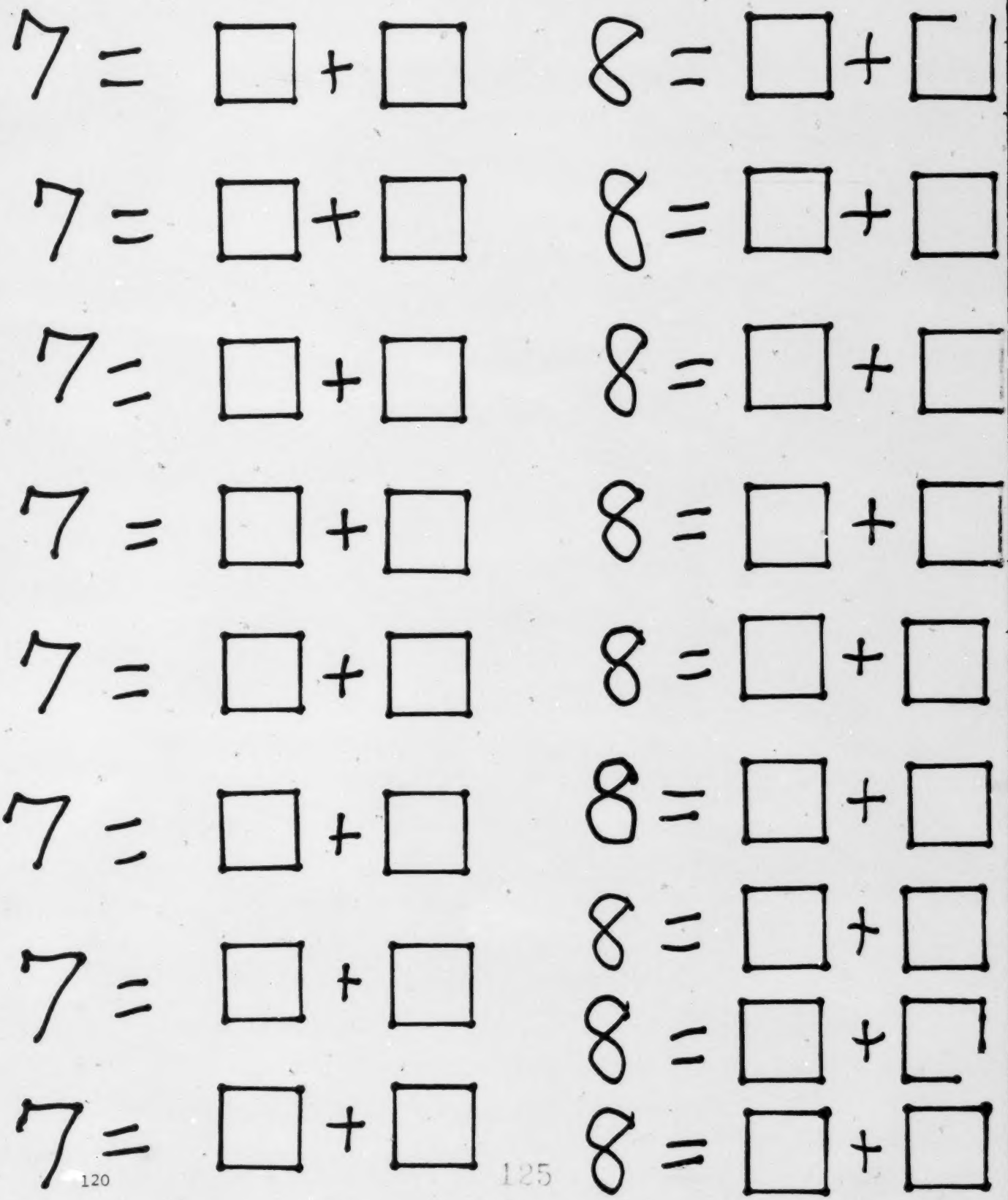




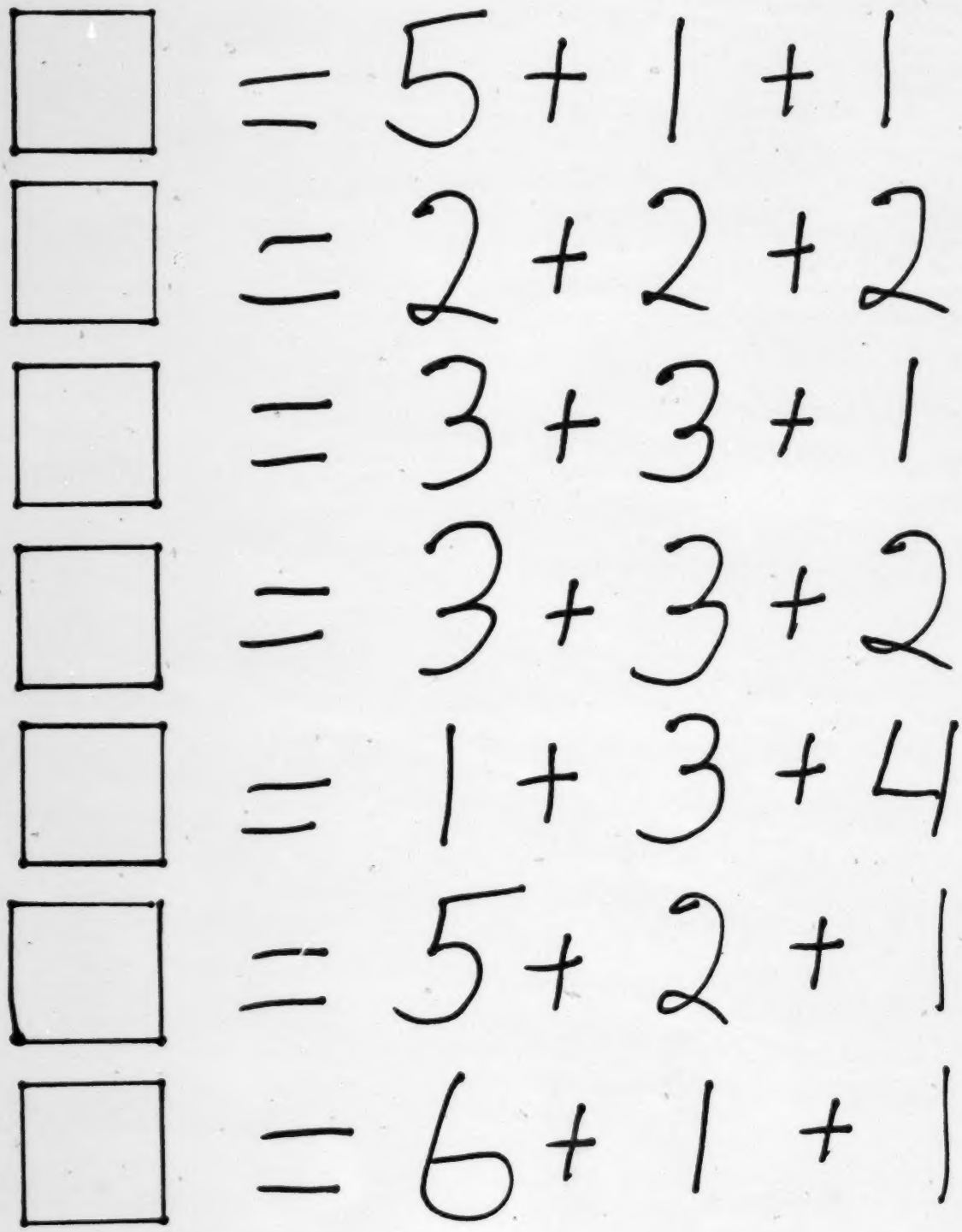
NAME _____



NAME _____



NAME_____



Introduction to Greater and Less.

Purpose(s): (a) To provide practice on constructing sets with more (fewer) members than a given set.

Materials: For each pupil: Balance; unmarked pans; 20 units.

Vocabulary: Put more (fewer) units on the other side.
How do you know that there are more (fewer) units?
5 is greater than 4.
3 is less than 5.

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Continue discussion until students verbalize 2 reasons: One based on the position of the balance and the other based on number. | PUT 3 UNITS ON THE L-SIDE.<br>PUT MORE UNITS ON THE R-SIDE.<br><br>HOW DO YOU KNOW THAT THERE ARE MORE UNITS ON THE R-SIDE?<br><br>HOW MANY ON L-SIDE?<br>HOW MANY ON R-SIDE?<br>( ____ ) IS GREATER THAN 3. |
| 2. Repeat (1) using 12 units on R-side. | |
| 3. Two reasons: positive of balance and number. . . . . | PUT 5 UNITS ON THE L-SIDE.<br>PUT FEWER UNITS ON THE R-SIDE.<br>HOW DO YOU KNOW THAT THERE ARE FEWER UNITS ON THE R-SIDE?<br>HOW MANY ON L-SIDE?<br>HOW MANY ON R-SIDE?<br>( _____ ) IS LESS THAN 5. |
| 4. Repeat (3) using 14 units on R-side. | |



End or Cont'd

## Introduction of Greater Than Sign

Purpose(s): (a) To introduce the greater than symbol.
(b) To provide practice on using the balance to show the greater than relation.

Materials: For each student: Balance; un-marked pans, greater than sign; 35 units.
Worksheets GL-4, 5.

Vocabulary:

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Show first problem on GL-4. . . . . | READ THIS SENTENCE. |
| | SHOW THIS ON YOUR BALANCE. |
| | ARE THE SIDES EQUAL? |
| Give each child | DO WE PUT AN EQUAL SIGN ON THE BALANCE? WHY? |
| Hold up | THIS IS A GREATER THAN SIGN.<br>PUT THIS SIGN ON YOUR BALANCE. |
| Point to second problem on GL-4. | WE READ: FIVE IS GREATER THAN THREE. |
| | WHAT DO WE WRITE IN THE CIRCLE TO SHOW THAT 5 IS GREATER THAN THREE? |
| | READ THE SENTENCE. |
| 2. Show third problem on GL-4. . . . . | READ THIS SENTENCE. |
| | SHOW ON YOUR BALANCE. |
| Point to fourth problem on GL-4. . . . . | WHAT SIGN DO WE PUT ON THE BALANCE?<br>WHAT DO WE WRITE IN THE CIRCLE?<br>READ THE SENTENCE. |
| 3. Have students complete GL-5. | |

End or Cont'd

NAME _____

5 is greater than 3

5 ◯ 3

13 is greater than 11

13 ◯ 11

Comparing Numbers

Purpose(s): (a) To provide practice on completing sentences using =, <, >.

Materials: For each student: Balance, un-marked pans; equals, greater than, and less than signs; 35 units.
Worksheets GL-8, 9.

Vocabulary:

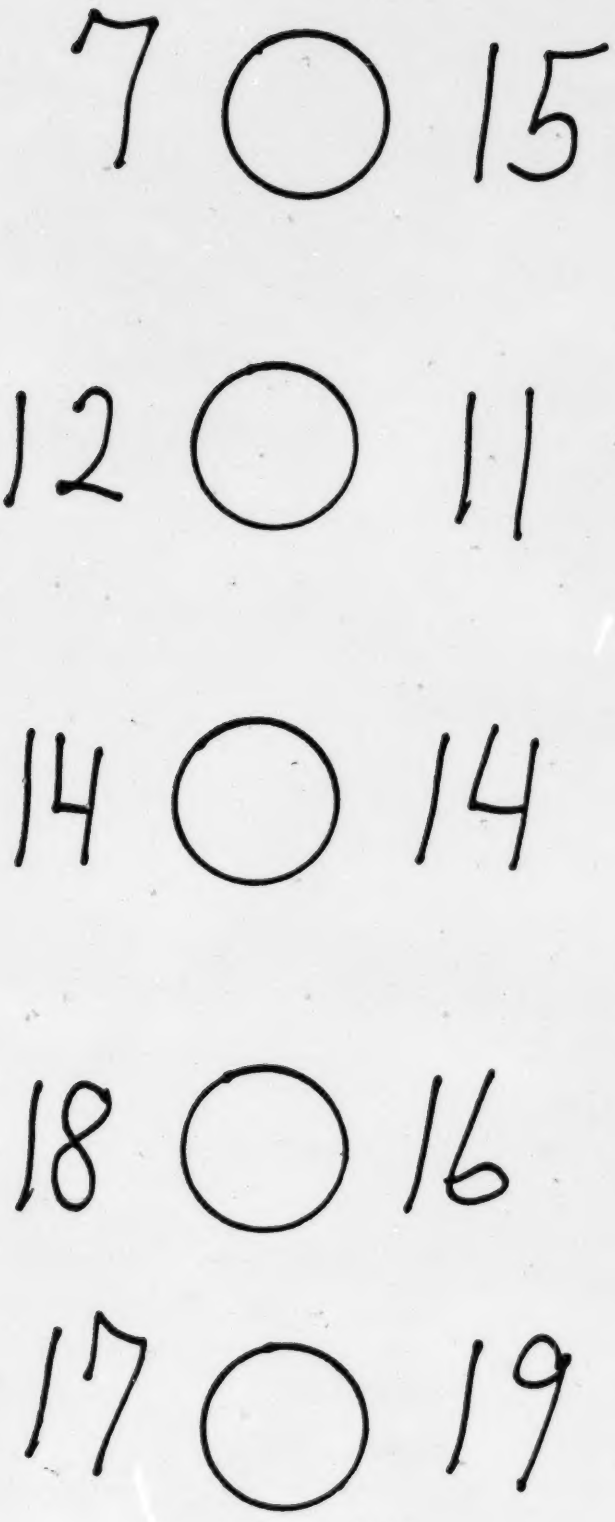
LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Show first problem on GL-8. . . . . | SHOW THIS ON YOUR BALANCE.<br>WHAT SIGN DO WE PUT ON THE BALANCE?<br>WHAT DO WE WRITE IN THE CIRCLE?<br>READ THE SENTENCE. |
| 2. Repeat (1) using other problems on GL-8. | |
| 3. Have students complete GL-9. | |



End or Cont'd

Name _____



Introduction of Longs

Purpose(s): (a) To establish that one ten is equivalent to ten ones.
(b) To establish that any two longs are equivalent.

Materials: For each student: Balance; unmarked pans; 35 units, 3 tens.

Vocabulary: One ten equals ten ones.

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Hold up a long..... | PUT ONE OF THESE ON THE LEFT SIDE. |
| Demonstrate. | |
| | IT EQUALS HOW MANY UNITS? |
| Point to long..... | THIS IS A TEN. |
| Show 1 ten ..... | READ THIS. |
| 2. | PUT 9 UNITS ON LEFT SIDE.<br>PUT 1 TEN ON RIGHT SIDE. |
| Demonstrate. | |
| | ARE THEY EQUAL?<br>HOW DO WE MAKE THEM EQUAL?<br>SEE 1 TEN IS ONE MORE THAN 9.<br>9 IS ONE LESS THAN 1 TEN. |
| 3. | PUT 11 UNITS ON LEFT SIDE.<br>PUT 1 TEN ON RIGHT SIDE.<br>ARE THEY EQUAL? |
| Follow student suggestions, but show taking one from eleven..... | HOW DO WE MAKE THEM EQUAL?<br>SEE 1 TEN IS ONE LESS THAN 11.<br>11 IS ONE MORE THAN A TEN. |



End or Cont'd

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 4. | PUT ONE TEN ON THE LEFT SIDE.<br>PUT ONE TEN ON THE RIGHT SIDE. |
| Demonstrate on balance. | ARE THEY EQUAL?<br>HOW DO WE KNOW? |



End or Cont'd

## Introduce Expanded Form

Purpose(s): (a) To provide drill on expressing a number 10-19, as the sum of 1 ten and a number less than 10.

Materials: For each student: Balance; unmarked pan left; 2-region pan right; 35 units, 3 tens.
Worksheets PV-4,5.

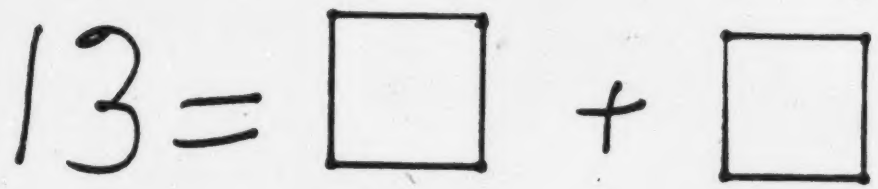
Vocabulary:

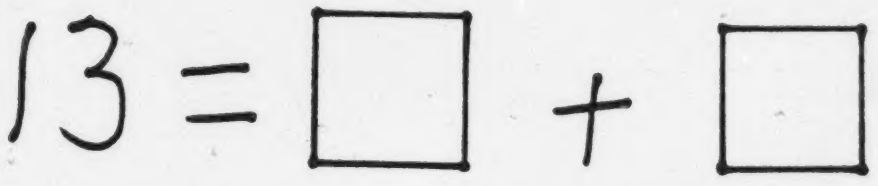
LESSON OUTLINE

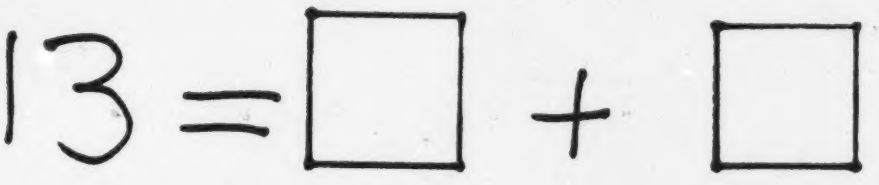
| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Show first problem on PV-4..... | READ THIS SENTENCE.<br>WHAT DOES IT TELL US TO DO ON THE BALANCE? |
| Follow student suggestions and complete first sentence. | |
| | IS THERE ANOTHER WAY TO MAKE IT EQUAL? |
| Follow student suggestions and complete second sentence. | |
| | IS THERE ANOTHER WAY TO MAKE IT EQUAL? |
| Follow student suggestion and complete third sentence. | |
| Stop, if 13 = 10 + 3 and 13 = 1 ten + 3 are shown. | |
| 2. Repeat (1) using remaining problems on PV-4. | |
| 3. Have students complete PV-5. | |

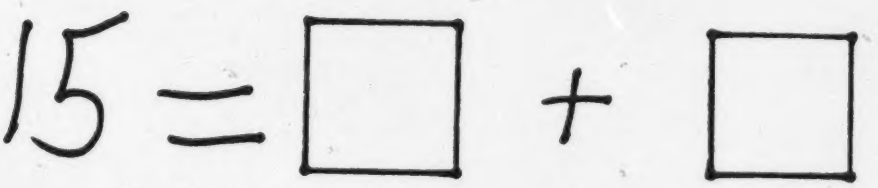
End or Cont'd

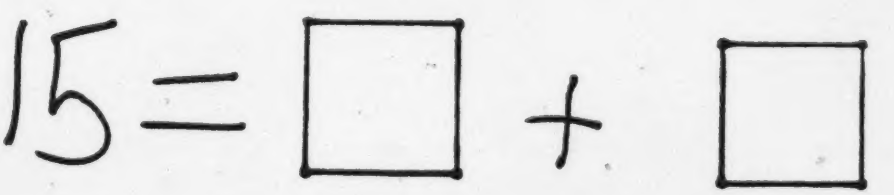
NAME _____

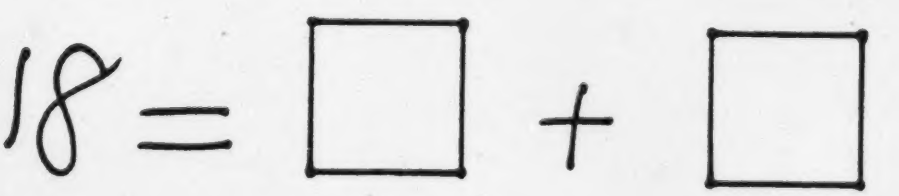


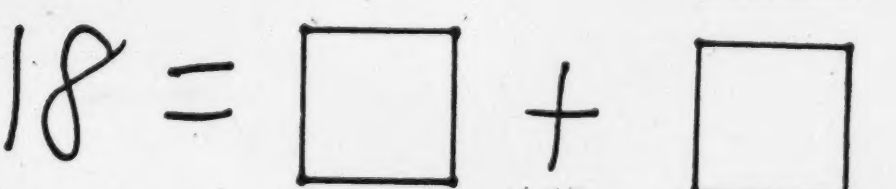












Relationships among addition facts

Purpose(s): (a) To teach relationships among addition problems which provide a basis for deriving a sum from a known fact.

Materials: Same as A-31.
Worksheets A 145-148

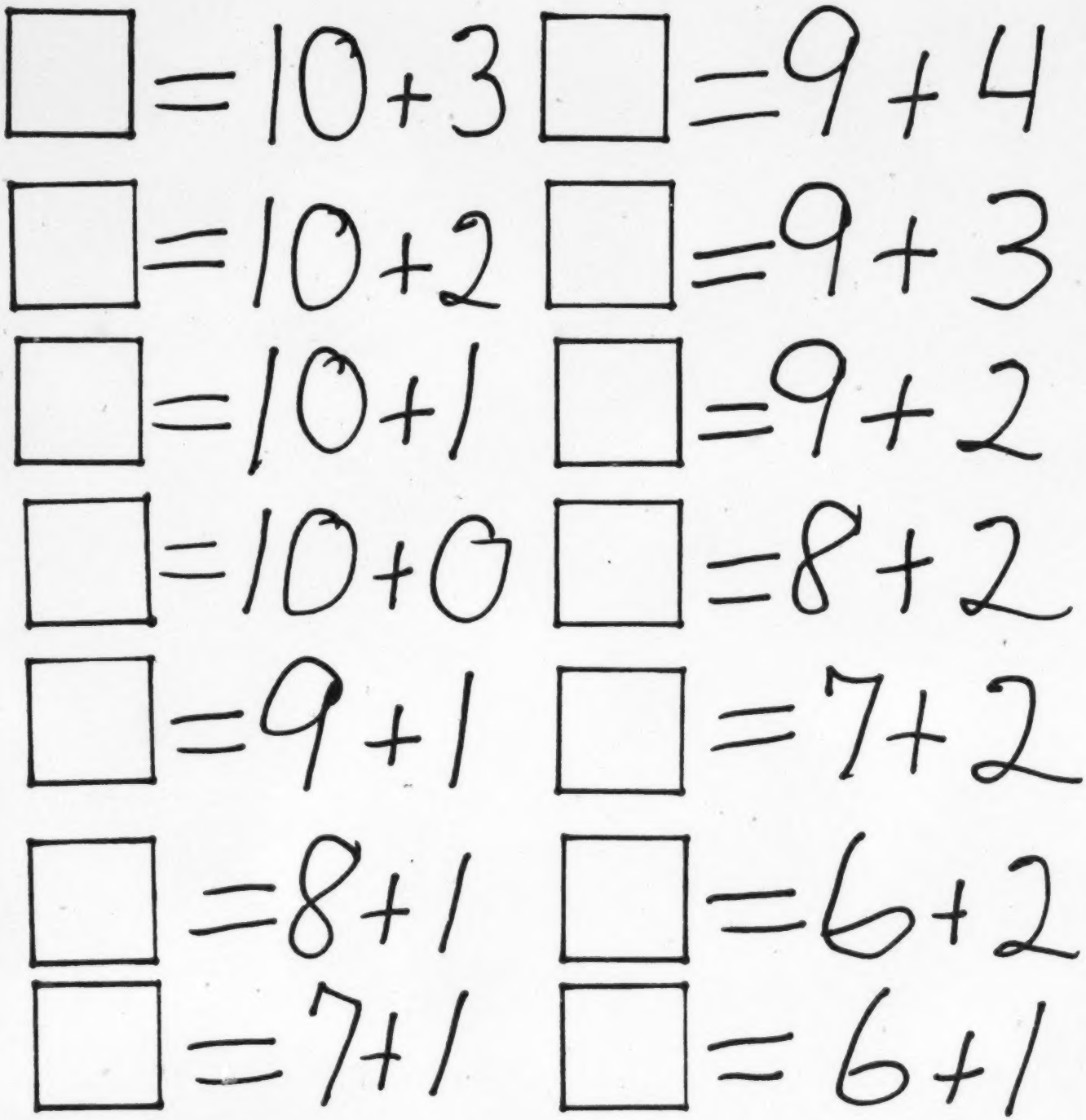
Vocabulary:

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Show problems in left column on A 145.<br><br>Demonstrate on balance. | READ THIS FOR ME.<br>WHAT DOES IT TELL US TO DO?<br>WHAT IS THE ANSWER?<br>WHAT DO WE WRITE IN THE BOX?<br>READ THE SENTENCE FOR ME.<br>DON'T TAKE UNITS OFF OF THE BALANCE. |
| 2. Point to second problem-left column A 145.<br><br>Demonstrate, take one unit from yellow. | READ THIS FOR ME.<br>HOW CAN WE CHANGE THE BALANCE TO SHOW THIS PROBLEM?<br><br>WHAT IS THE ANSWER?<br>WRITE IT IN THE BOX?<br>READ THE SENTENCE FOR ME.<br>DON'T TAKE UNITS OFF OF THE BALANCE. |
| 3. Continue (2) with remaining problems in left column A 145. | |
| 4. Have students complete A 145-148. | LOOK FOR WAYS OF CHANGING THE BALANCE TO MAKE NEW PROBLEMS. |

End or Cont'd

NAME _____



Introduce a + b=c + d

Purpose(s): (a) To teach relationships among addition facts.

Materials: For each student: Balance; 2-region pans on both sides, 35 units; 2 tens. Worksheets A 149-152.

Vocabulary:

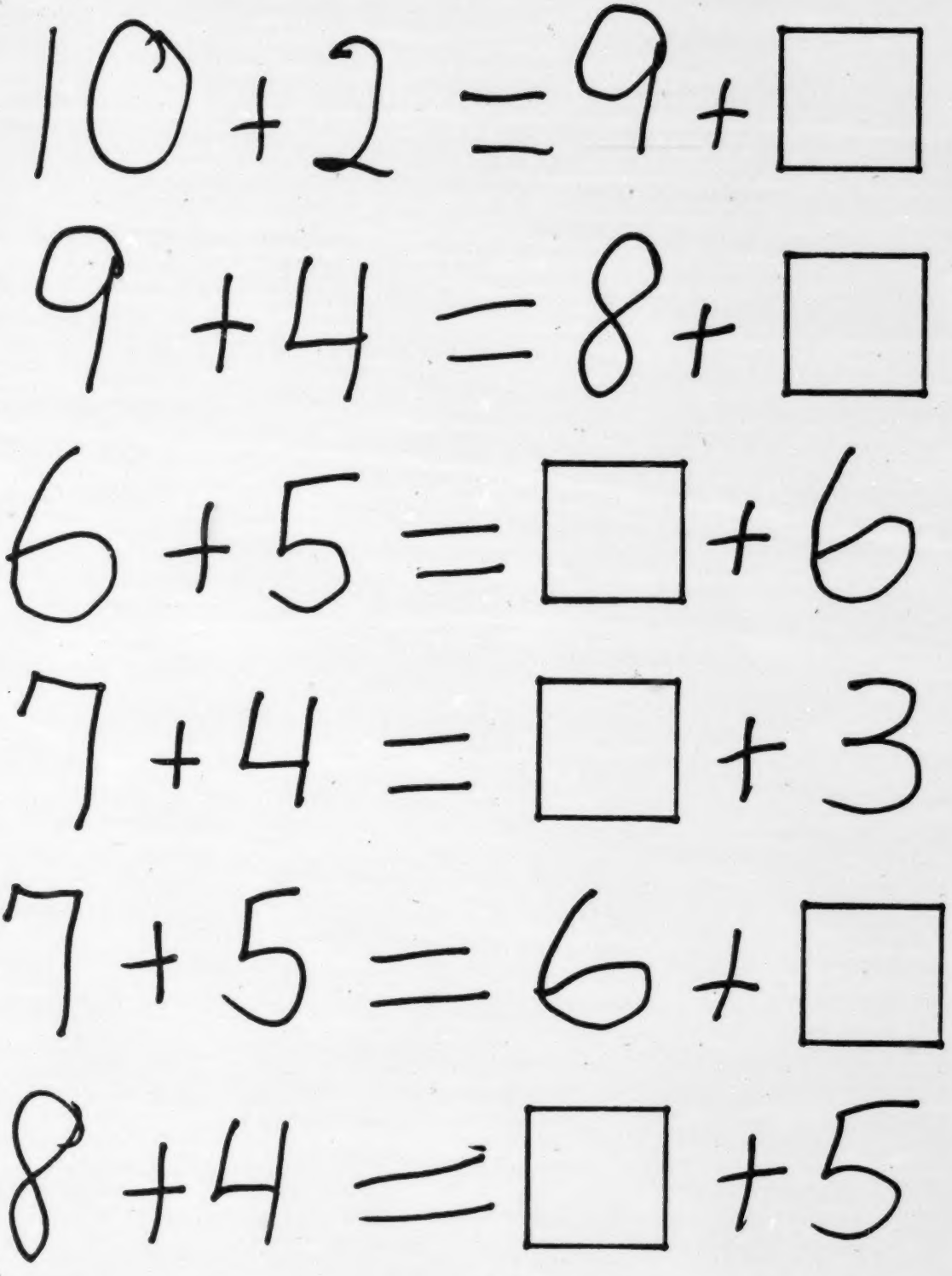
LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Show first problem on A 149.<br><br>Demonstrate on balance.<br><br><br>Demonstrate on balance. | READ THIS SENTENCE FOR ME.<br>WHAT DOES IT TELL US TO DO?<br><br>HOW MANY UNITS DO WE PUT ON THE R-SIDE TO MAKE IT EQUAL.<br><br>CAN YOU THINK OF AN EASY WAY TO FIND OUT?<br><br>WRITE THE ANSWER IN THE BOX.<br>READ THIS FOR ME. |
| 2. Use procedures in (1) to do second problem in A 149. | |
| 3. Have students complete A 149. | |
| 4. Have students complete A 150-152. | |

133

End or Cont'd

NAME_____



Comparing sums.

Purpose(s): (a) To provide practice on completing sentences using $=, <, >$.

Materials: Same as A-35.
Worksheets GL 14-15

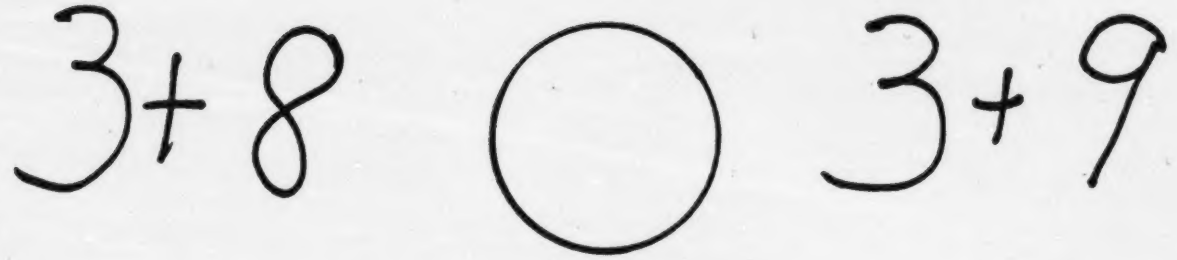
Vocabulary:

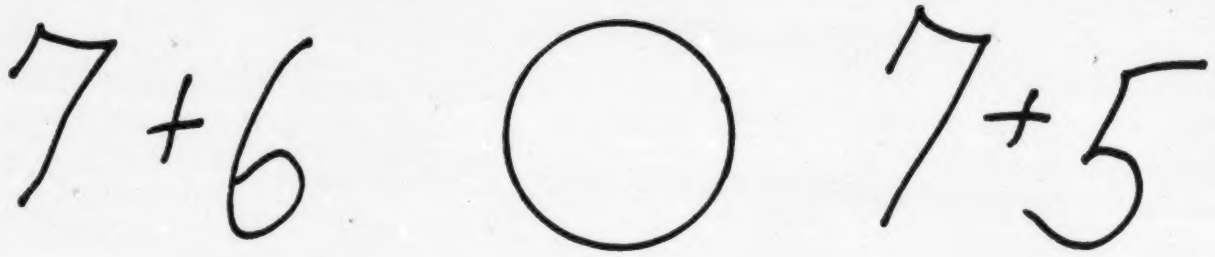
LESSON OUTLINE

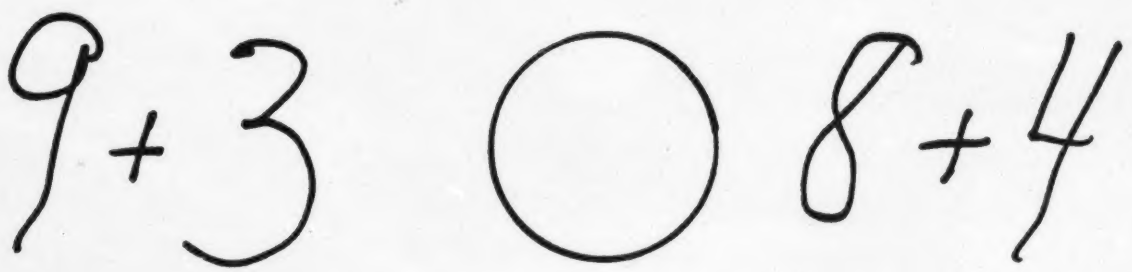
| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Show first problem on GL-14<br><br>Demonstrate. | WHAT DOES THIS PROBLEM TELL US TO DO?<br><br>WHAT SIGN DO WE PUT ON THE BALANCE<br>WHAT DO WE WRITE IN THE CIRCLE?<br>READ THE SENTENCE FOR ME. |
| 2. Use procedures in (1) above to complete GL-14. | |
| 3. Have students complete GL 15. | |
| | |

End or Cont'd

NAME _____



Numbers 20-29.

Purpose(s): (a) To teach the expanded form of the numbers 20-29.

Materials: For each child: Balance; unmarked pan Left, 2 regions pan right; 35 units; 3 tens.
Worksheets PV 9-11.

Vocabulary:

LESSON OUTLINE

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
|---|---|
| 1. Demonstrate by putting 2 tens. | PUT 20 UNITS ON THE L-SIDE.<br>PUT SOMETHING ON THE R-SIDE TO MAKE IT EQUAL.<br>HOW MANY ON EACH SIDE. |
| 2. Demonstrate. | PUT ONE MORE UNIT ON THE L-SIDE.<br>HOW MANY UNITS ON L-SIDE.<br>PUT SOMETHING ON THE R-SIDE TO MAKE IT EQUAL.<br>HOW MANY ON EACH SIDE. |
| 3. Continue (2) with numbers through 29. | |
| 4. Show first problem on PV-9.<br>Demonstrate. | READ THIS FOR ME.<br>WHAT DOES IT TELL US TO DO?<br>WHAT DO WE WRITE IN THE BOXES.<br>READ THE SENTENCE FOR ME. |
| 5. Continue (4) with second problem on PV 9. | |



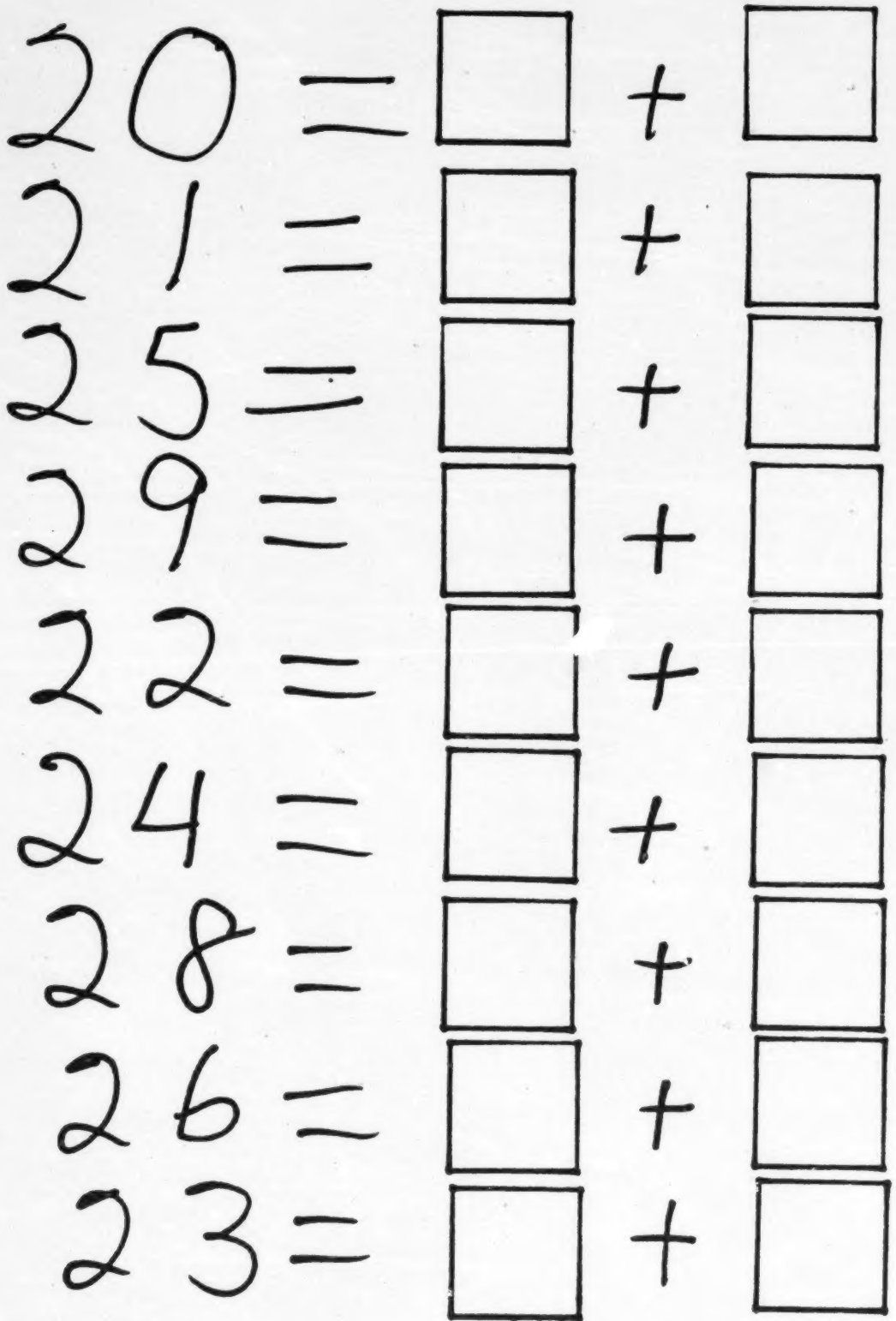
End or Cont'd

| ACTION | QUESTIONS, DIRECTIONS, ETC. |
| --- | --- |
| 6. Have students complete PV 9 and PV 10-11. | |



End or Cont'd

NAME _____



APPENDIX C

INTERNAL EVALUATION POSTTEST

Part I.

Directions: WORK THESE PROBLEMS.

A. $3 + 1 = \square$

$\begin{array}{r} 4 \\ +2 \\ \hline \end{array}$

$2 + 2 = \square$

$\begin{array}{r} 1 \\ +1 \\ \hline \end{array}$

$1 + 6 = \square$

$\begin{array}{r} 3 \\ +5 \\ \hline \end{array}$

$5 + 2 = \square$

$\begin{array}{r} 1 \\ +7 \\ \hline \end{array}$

$7 + 2 = \square$

$\begin{array}{r} 7 \\ +1 \\ \hline \end{array}$

$0 + 1 = \square$

$\begin{array}{r} 6 \\ +1 \\ \hline \end{array}$

$7 + 0 = \square$

$\begin{array}{r} 5 \\ +4 \\ \hline \end{array}$

$4 + 3 = \square$

B. $4 - 2 = \square$

$\begin{array}{r} 8 \\ -1 \\ \hline \end{array}$

$3 - 1 = \square$

$\begin{array}{r} 6 \\ -0 \\ \hline \end{array}$

$3 - 0 = \square$

$\begin{array}{r} 8 \\ -7 \\ \hline \end{array}$

$7 - 5 = \square$

$\begin{array}{r} 7 \\ -5 \\ \hline \end{array}$

$2 - 2 = \square$

$\begin{array}{r} 6 \\ -3 \\ \hline \end{array}$

$5 - 2 = \square$

$\begin{array}{r} 6 \\ -6 \\ \hline \end{array}$

$8 - 4 = \square$

$\begin{array}{r} 6 \\ -2 \\ \hline \end{array}$

$\begin{array}{r} 7 \\ -2 \\ \hline \end{array}$

C. $9 + 3 = \square$

$\begin{array}{r} 8 \\ +3 \\ \hline \end{array}$

$\begin{array}{r} 7 \\ +6 \\ \hline \end{array}$

$7 + 7 = \square$

$\begin{array}{r} 6 \\ +6 \\ \hline \end{array}$

$8 + 5 = \square$

Part II.

A. Balance: No pans on balance, but two 1-region and two 2-region pans beside the balance.

Place the following card in front of the student: $5 + 4 = \square$

Say: READ THIS FOR ME.

SHOW IT ON THE BALANCE.

WHAT DO WE WRITE IN THE BOX? (Allow student to write answer.)

READ IT FOR ME.

Remove pans from balance.

Repeat above with the following cards:

$\square - 9 - 3$ $\quad 8 = \square + 3$ $\quad 6 - 6 = \square$ $\quad 7 + 5 = 5 + \square$

B. Balance: 1-region pans on both sides

Place the following card in front of the student: $\boxed{24}$

Say: PUT THIS MANY ON THE L-SIDE.

HOW MANY?

PUT MORE ON THE R-SIDE.

HOW MANY?

C. Set up balance to show $5 = 3 + 2$.

Say: WRITE A SENTENCE TO SHOW WHAT IS ON THE BALANCE.

READ YOUR SENTENCE.

WHY IS IT EQUAL (NOT EQUAL, MORE, LESS)?

Repeat above using the following problems:

$4 = 4$ $\quad 3 \neq 5$

Part III.

A. Balance: No pans on balance, but two 1-region pans and two 2-region pans beside the balance.

Place the following card in front of the student: $8 \bigcirc 5 + 3$ .

Say: WRITE SOMETHING IN THE CIRCLE TO MAKE IT RIGHT.

HOW DO YOU KNOW THAT IT IS RIGHT?

Remove pans from balance.

Repeat using the following cards:

$7 + 2 \bigcirc 9$ $12 \bigcirc 5 + 5$ $3 + 5 \bigcirc 7$

B. Balance: 1-region pan left; 2-region pan right.

Place the following card in front of the student: $\square$ = 2 tens + 8

Say: SHOW THIS ON THE BALANCE.

WHAT DO WE WRITE IN THE BOX? (Allow student to write answer.)

READ IT FOR ME.

Leave balance set up.

Place the following card in front of the student: $\square = 20 + 8$

Say: SHOW THIS ON THE BALANCE.

WHAT DO WE WRITE IN THE BOX? (Allow student to write answer.)

READ IT FOR ME.

C. Set up balance to show $12 = 7 + 5$.

Say: WHAT DOES THE BALANCE SHOW?

Leave balance set up.

Place the following card in front of student: $11 = 6 + 5$

Say: SHOW THIS ON THE BALANCE.

Leave balance set up.

Place the following card in front of student: $11 = 5 + 6$

Say: SHOW THIS ON THE BALANCE.

Part IV.

A. Balance: No pans on balance, but two 1-region and two 2-region pans beside the balance.

Place the following card in front of the student: $\square = 3 + 7$

Say: READ THIS FOR ME.

SHOW IT ON THE BALANCE.

WHAT DO WE WRITE IN THE BOX. (Allow student to write answer.)

READ IT FOR ME.

Remove pans from balance.

Repeat above with the following cards:

$8 - 5 = \square$ $9 = \square + \square$ $\square + 4 = 7$ $\square = 6 - 0$

B. Balance: 1-region pans on both sides.

Place the following card in front of the student: 18

Say: PUT THIS MANY ON THE L-SIDE.

HOW MANY?

PUT LESS ON THE R-SIDE.

HOW MANY?

C. Set up balance to show $10 = 6 + 4$

Say: WHAT DOES THE BALANCE SHOW?

Leave balance set up.

Place the following card in front of student: $11 = \square + \square$

Say: SHOW THIS ON THE BALANCE.

FILL IN THE BOXES.

D. Place 2 dimes and 3 pennies in front of student.

Say: HOW MUCH MONEY?

Leave money on table.

146 Place 1 dime and 7 pennies in front of student.

Part IV. (continued)

D. (continued)

Say: HOW MUCH MONEY?

Leave money on table.

Point to both piles of money.

Say: WHICH PILE HAS MORE MONEY?

HOW DO YOU KNOW?

# APPENDIX D

## EXTERNAL EVALUATION: TEST I

1. (a). Show card: 3 = □ + □

   Say: READ THIS FOR ME.

   WRITE SOMETHING TO MAKE IT OKAY.

   READ THIS FOR ME.

   IS IT OKAY?

   (b). Repeat (a) three more times, saying: WRITE SOMETHING DIFFERENT TO MAKE IT OKAY.

2. Show card: □ = 4 + 2

   Say: WRITE SOMETHING TO MAKE IT OKAY.

   READ IT FOR ME.

   IS IT OKAY?

3. Repeat 2, showing card: 5◯3 + 2

4. Repeat 2, showing card: 3◯1 + 1

5. Repeat 2, showing card: 4◯3 + 2

6. Repeat 1, showing card: □ + □ = 3

7. Repeat 2, showing card: 2 + 3 = □

8. Repeat 2, showing card: 1 + 5◯6

9. Repeat 2, showing card: 3 + 1◯5

10. Repeat 2, showing card: 2 + 3◯4

11. Show card: 3 = 3

    Say: IS IT OKAY?

12. Repeat 11, showing card: 6 = 2 + 4

13. Repeat 11, showing card: 3 + 4 = 5 + 2

14. Repeat 11, showing card: 2 + 3 = 5 + 1

# APPENDIX E

## EXTERNAL EVALUATION: TEST II

1. Place the following aids on table: beans, sticks, unifix cubes, and blocks.

   Show card: 5 + 3 = ☐

   Say: WORK THIS PROBLEM FOR ME. (Student writes answer in box.)

   USE ANY ONE OF THESE (pointing to aids) TO SHOW ME WHAT THIS PROBLEM SAYS.

2. Repeat 1, showing card: 9 - 4 = ☐

3. Repeat 1, showing card: ☐ = 4 + 2

4. Repeat 1, showing card: ☐ = 6 - 1

5. Repeat 1, showing card: 5 = ☐ + 2

6. Repeat 1, showing card: 7 = 1 + ☐

7. Repeat 1, showing card: ☐ + 3 = 4

8. Repeat 1, showing card: 2 + ☐ = 6

9. Repeat 1, showing card: 7 = ☐ + ☐

10. Repeat 1, showing card: 4 = ☐ + ☐

11. Repeat 1, showing card: ☐ + ☐ = 7

12. Repeat 1, showing card: ☐ - ☐ = 4

13. Repeat 1, showing card: 3 + 4 = ☐ + ☐

14. Repeat 1, show card: 3 + 4 = ☐ + ☐

    Say: DO THIS ONE IN A DIFFERENT WAY.



155

# APPENDIX F

## EXTERNAL EVALUATION: TEST III

$T_3$ Final Interview June 1976

Statements (1) - (6): Interviewer writes in front of the child. Each statement is followed by the same 4 inquiries.

(1) 3 = 3
- (a) Read this.
- (b) Is this okay?
- (c) What does it mean?
- (d) Show me this with the beans.

(2) 7 = 5 + 2 ←
(3) 3 + 4 = 7 ← reverse the order for the control group

(4) 1 + 2 = 2 + 1

(5) 3 + 2 = 4 + 1

(6) 1 + 3 = 4 + 2

Statements (7) and (8) are presented on cards, followed by the same 3 inquiries

(7) 13 = 10 + 3
- (a) Read this.
- (b) Is this okay?
- (c) What does it mean?

(8) 10 + 5 = 15

Statements (9) - (11) are given orally. Say: I'll tell you something and you tell me whether it's okay.

(9) 5 = 4 + 1 ←
(10) 2 + 4 = 6 ← reverse the order for the control group

(11) 3 + 2 = 4 + 1

(12) Show a card: 3 + ☐ = 5
Say: What goes in the box to make it okay?

(13) Show a card: 6 = 4 + ☐
Say: What goes in the box to make it okay?

(14) Oral: I have three pencils. You have some pencils. Together we have five pencils. How many pencils do you have?

BIBLIOGRAPHY

Anderson, W.C. The development and evaluation of a unit of instruction designed to teach second grade children the concept of mathematical equality. Unpublished doctoral dissertation, Florida State University, 1976.

Beattie, I.D, & Deichmann, J.W. Error trends in solving number sentences in relation to workbook format across first and second grades. Paper presented at the American Educational Research Association Annual Meeting, Chicago, April 1972. (ERIC Document Reproduction Service, No. ED 064 170)

Behr, M., Erlwanger, S., & Nichols, E. How children view equality sentences. PMDC Technical Report No. 3, Tallahassee, Florida, 1977.

Frazer, C.D. Abilities of college students to involve symmetry of equality with applications of mathematical generalizations. Unpublished doctoral dissertation, Florida State University, 1976.

Grouws, D.A. Some factors associated with children's solving performance on four types of mathematical open sentences. Paper presented at the annual meeting of the American Educational Research Association, Chicago, April 1972. (ERIC Document Reproduction Service No. ED 061 105)

Renwick, E.M. Children's misconceptions concerning the symbol for mathematical equality. British Journal of Educational Psychology, 1932, II (Part 1), 173-183.

Weaver, J.F. The symmetric property of the equality relation and young children's ability to solve open addition and subtraction sentences. Journal for Research in Mathematics Education, 1973, 4 (1), 45-56.